A GENERALIZATION OF STUDY'S THEOREM
ON CONVEX MAPS

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ABSTRACT. Study proved that disks of radius r, $0 < r < 1$, get mapped onto convex (starlike) sets under all convex (starlike) maps of the unit disk U.

Pommerenke and Heins gave a geometric characterization of the subsets of U that get mapped onto convex sets under all convex maps. In this note we give an analytical characterization of the subsets of U that get mapped onto α-starlike domains under all α-starlike maps of U for all $\alpha \geqslant 0$ noting that 0-starlikeness equals starlikeness and 1-starlikeness equals convexity.

Introduction. For any number r, let $U_r = \{z: |z| < r\}$ and $U = U_1$. Let S denote the collection of functions $f(z) = z + \ldots$ that are regular and univalent in U. We let $S^* = \{f: f \in S$ and $f(U)$ is starlike$\}$. Whenever we say a domain is starlike we mean that it is starlike with respect to the origin. Let $K = \{f: f \in S$ and $f(U)$ is convex$\}$. The requirement that $f'(0) = 1$ is not essential to the arguments in this paper. A subclass of S defined by Mocanu in [6] gives a continuous parametrization between the starlike functions and the convex functions. For any fixed α, $\alpha \geqslant 0$, this subclass is given by

$$S^*(\alpha) = \Big\{ f: f(z) f'(z)/z \neq 0,$$
$$\operatorname{Re}\big\{ \alpha\big[1 + zf''(z)/f'(z)\big] + (1 - \alpha)\big[zf'(z)/f(z)\big]\big\} > 0\Big\}.$$

We shall call functions in $S^*(\alpha)$ α-starlike functions. A number of authors [7] have shown that $S^*(\alpha)$ is a subclass of S^* for $\alpha \geqslant 0$. We note that $S^*(0) = S^*$ and $S^*(1) = K$. A domain D will be called α-starlike if and only if for every univalent function f for which $f(U) = D$ we have $b(f)$ is in $S^*(\alpha)$ for some positive scalar b. We define for a fixed α, $\alpha \geqslant 0$, the set $H_{S^*(\alpha)} = \{X \subset U: f(X)$ is an α-starlike domain under all α-starlike maps f of $U\}$. The cases $\alpha = 0$ and $\alpha = 1$ have been investigated by many authors.

In [9], Study showed that for any r, $0 < r \leqslant 1$, the disk U_r is in H_{S^*} and in H_K. In [1], [2] Alexandrov characterized those subsets of U that are mapped onto convex domains as well as those that are mapped onto starlike domains under all regular univalent maps of U. Problem 6.20 in Hayman's book [3] asks the slightly more general question of characterizing members of H_K but allows meromorphic convex maps of U. M. Reade proposed the problem of characterizing the sets of H_K to C. Pommerenke, who subsequently gave a

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solution to the problem in [8]. Later Heins [4], gave an alternate character-
ization of sets in H_K including a topological representation for these sets. He
continued this investigation in [5]. To discuss Pommerenke's and Heins's
characterization we need to recall the notion of D-convexity (the name is due
to Heins). Given two distinct points a, b in U there are exactly two circles C_0
and C_1 called oricycles which lie in the closure of U, pass through both a and
b, and have a tangent in common with the unit circle. We denote by $D(a,b)$
the intersection of the disks determined by C_0 and C_1. A subset X of U is said
to be D-convex if $D(a,b)$ is contained in X whenever a and b are distinct
points of X. Pommerenke and Heins proved that X is in H_K if and only if X is
D-convex. In this paper we characterize the sets in $H_{S^*(\alpha)}$ for any fixed $\alpha \geqslant 0$,
i.e. we characterize the subsets of U that get mapped onto α-starlike domains
under all α-starlike maps of U. Although this characterization does not have
the particularly elegant geometric formulation that Heins's work has, this
seems to be in the nature of things with respect to the more general class
$S^*(\alpha)$. However, for the case $\alpha = 1$ (our) Theorem 2 does give an analytical
characterization of the sets described geometrically by Pommerenke and
Heins.

REMARK. The author has recently been informed that for the starlike case
of $\alpha = 0$ a very interesting characterization of the sets in H_{S^*}, using a
generalization of Heins's concept of D-convexity, has just been obtained by
Peter Fenton.

Characterization of H_{S^*}. We first consider the simpler case when $\alpha = 0$. Let
X be a subset of U. Since the identity map is in S^* and since X is starlike with
respect to the origin we may assume X is a starlike set containing the origin.
The Riemann Mapping Theorem assures the existence of an univalent
analytic function $w = w(z)$ on U such that $w(U) = X$, $w(0) = 0$ and $w'(0) >
0$. Throughout this paper we will be using the principal branch of the log to
define the 'arg' function. With this notation we shall characterize the mem-
bers of H_{S^*} in the following theorem.

THEOREM 1. *Let X be a subset of U with the Riemann map w, $w(U) = X$,*
$w(0) = 0$ and $w'(0) > 0$. Then X is in H_{S^*} if and only if

$$\left| \arg \frac{zw'(z)}{w(z)} \right| \leqslant \cos^{-1} \frac{2|w(z)|}{1 + |w(z)|^2}, \qquad z \in U. \qquad (1)$$

PROOF. We first show that (1) implies X is in H_{S^*}. We must show for any f
in S^* that $f(X)$ is starlike. If we define g on U by $g(z) = f[w(z)]$, where $w(z)$
is the above Riemann map, then we only have to show that g is in S^*. Since f
is in S^* there exists a Schwarz function ϕ, analytic on U, $\phi(0) = 0$ and
$|\phi(w)| \leqslant |w|$, such that $wf'(w)/f(w) = (1 + \phi(w))/(1 - \phi(w))$. Thus

$$\left| \arg \frac{zg'(z)}{g(z)} \right| = \left| \arg \frac{zf'(w)w'(z)}{f(w)} \right| \leqslant \left| \arg \frac{zw'}{w} \right| + \left| \arg \frac{1 + \phi(w)}{1 - \phi(w)} \right|. \quad (2)$$

Since $|\phi(w)| \leqslant |w| < 1$, $(1 + \phi)/(1 - \phi)$ is contained in the disk of radius $2|\phi|/(1 - |\phi|^2)$ with center $(1 + |\phi|^2)/(1 - |\phi|^2)$. Thus the right-hand side of (2) is less than or equal to

$$\left| \arg \frac{zw'}{w} \right| + \sin^{-1}\left(\frac{2|\phi|}{1 + |\phi|^2} \right) \leqslant \left| \arg \frac{zw'}{w} \right| + \sin^{-1}\left(\frac{2|w|}{1 + |w|^2} \right)$$

which is less than or equal to $\pi/2$ whenever

$$\left| \arg \frac{zw'(z)}{w(z)} \right| \leqslant \frac{\pi}{2} - \sin^{-1}\left(\frac{2|w|}{1 + |w|^2} \right) = \cos^{-1}\left(\frac{2|w|}{1 + |w|^2} \right).$$

Thus if (1) holds then $g(U) = f(X)$ is starlike for each $f \in S^*$.

For the converse assume there exists a point w_0 in X such that $w_0 = w(z_0)$ where $|\arg(z_0 w'(z_0)/w(z_0))| > \cos^{-1} 2|w(z_0)|/(1 + |w(z_0)|^2)$. Since X is at least starlike, then by a rotation and or reflection of X we can assume $w_0 > 0$ and

$$0 \leqslant \arg[z_0 w'(z_0)/w(z_0)] < \pi/2.$$

By choosing $f_0(z) = z(1 - iz)^{-2}$ in S^* we have

$$\left| \arg \frac{z_0 g'(z_0)}{g(z_0)} \right| = \arg \frac{z_0 w'(z_0)}{w(z_0)} + \arg \frac{1 + iw_0}{1 - iw_0}$$

$$> \cos^{-1}\left(\frac{2w_0}{1 + w_0^2} \right) + \sin^{-1}\left(\frac{2w_0}{1 + w_0^2} \right) = \frac{\pi}{2}.$$

Thus, it follows that $b(f)$ is not in S^* for any scalars. This completes the proof of Theorem 1.

Characterization of $H_{S^*(\alpha)}$. In order to do the more general case of $\alpha \geqslant 0$, a bound is needed for the argument of a sum rather than the argument of a product as in (2). Thus a different approach is required. We present some preliminary notation. Given a function h, analytic on U, $h(0) = 0$, $h(z)h'(z)/z \neq 0$, we define for a given $\alpha \geqslant 0$ and r, $0 < r < 1$,

$$J_h(\theta) = (1 - \alpha)\arg h(re^{i\theta}) + \alpha\left[\arg(re^{i\theta})h'(re^{i\theta}) \right]. \quad (3)$$

Since for each $\alpha \geqslant 0$ $f(z) = z$ is in $S^*(\alpha)$, we may assume that the set X in $H_{S^*(\alpha)}$ is at least starlike and contains the origin. Let w be the Riemann map such that $w(U) = X$, $w(0) = 0$, and $w'(0) > 0$. For any r, $0 < r < 1$, let $w_0 = w(re^{i\theta_0})$ and $w_1 = w(re^{i\theta_1})$ be two distinct points in X. The starlikeness of X assures that $w(z)w'(z)/z \neq 0$ for z in U. We may assume $0 < \arg(w_1/w_0) \leqslant \pi$ since X is starlike $r < 1$, and

$$\mathrm{Re}\{ zw'(z)/w(z) \} = \frac{\partial}{\partial\theta}\left[\arg w(re^{i\theta}) \right] > 0,$$

so that arg $w_0 \neq \arg w_1$. Thus $0 \leqslant \theta_0 < \theta_1 < 2\pi$. Let the two uniquely determined oricycles passing through w_0 and w_1 be denoted by C_0 and C_1, with centers η_0, η_1 and the common points of tangency to the unit circle by ζ_0, ζ_1, respectively. One of the common points of tangency say ζ_0 is such that

$$0 < \arg \frac{w_1 - \zeta_0}{w_0 - \zeta_0} < \pi. \tag{4}$$

From a geometric property of a circle we have that

$$2\arg \frac{w_1 - \zeta_0}{w_0 - \zeta_0} = \arg \frac{w_1 - \eta_0}{w_0 - \eta_0}$$

when $0 < \arg[w_1/w_0] \leqslant \pi$.

With this notation we will characterize the members of $H_{S^*(\alpha)}$ in the following theorem.

THEOREM 2. *Given any nonnegative α, X is in $H_{S^*(\alpha)}$ if and only if for each r, $0 < r < 1$, and $0 \leqslant \theta_0 < \theta_1 < 2\pi$ such that*

$$0 < \arg\left[w(re^{i\theta_1})/w(re^{i\theta_0}) \right] \leqslant \pi$$

we have

$$J_w(\theta_1) - J_w(\theta_0) \geqslant \arg \frac{w(re^{i\theta_1}) - \eta_0}{w(re^{i\theta_0}) - \eta_0} \tag{5}$$

with $J_w(\theta)$ defined by (3).

PROOF. We shall first prove the sufficiency of condition (5). Let f be in $S^*(\alpha)$ and define $g(z) = f[w(z)]$. Then we need to prove that $g(U)$ is α-starlike, i.e., we need to show that

$$\mathrm{Re}\left\{ \alpha\left[1 + \frac{zg''(z)}{g'(z)} \right] + (1 - \alpha)\left[\frac{zg'(z)}{g(z)} \right] \right\} = \frac{\partial J_g(\theta)}{\partial \theta}$$

is nonnegative in $|z| \leqslant r < 1$. Since $\partial J_g(\theta)/\partial\theta$ is harmonic it suffices by the minimum principle to consider only $|z| = r$.

Using the definition of g we obtain

$$\frac{zg'(z)}{g(z)} = \frac{wf'(w)}{f(w)} \cdot \frac{zw'(z)}{w(z)} \tag{6}$$

and

$$1 + \frac{zg''(z)}{g'(z)} = \frac{zw'(z)}{w(z)} \cdot \frac{wf''(w)}{f'(w)} + 1 + \frac{zw''(z)}{w'(z)}.$$

Thus

$$\frac{\partial J_g(\theta)}{\partial \theta} = \text{Re}\left\{ \alpha \left[1 + \frac{zw''(z)}{w'(z)} + \frac{zw'(z)}{w(z)} \cdot \frac{wf''(w)}{f'(w)} \right]\right.$$

$$\left.+ (1-\alpha)\left[\frac{zw'(z)}{w(z)} \cdot \frac{wf'(w)}{f(w)} \right] \right\}$$

$$= \text{Re}\left\{ \alpha \left[1 + \frac{zw''(z)}{w'(z)} + \frac{zw'(z)}{w(z)} \left[\alpha \frac{wf''(w)}{f'(w)} \right] + (1-\alpha) \frac{wf'(w)}{f(w)} \right] \right\}$$

$$= \text{Re}\left\{ \alpha \left[1 + \frac{zw''(z)}{w'(z)} \right] - \alpha \frac{zw'(z)}{w(z)} \right.$$

$$\left. + \frac{zw'(z)}{w(z)} \left[\alpha \left(1 + \frac{wf''(w)}{f'(w)} \right) + (1-\alpha)\frac{wf'(w)}{f(w)} \right] \right\}$$

$$= \text{Re}\left\{ \alpha \left[1 + \frac{zw''(z)}{w'(z)} \right] - \alpha \frac{zw'(z)}{w(z)} + \frac{zw'(z)}{w(z)} \left[\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}+w}{e^{it}-w} du(t) \right] \right\}, \quad (7)$$

where the existence in (7) of a nonnegative, nondecreasing measure u on $[0, 2\pi]$ with $u(2\pi) - u(0) = 2\pi$ is guaranteed since f is in $S^*(\alpha)$.

Letting $z = re^{i\theta}$ we see that the nonnegativeness of (7) is equivalent to the nonnegativeness of the following quantity for all θ_0, θ_1, $0 \leqslant \theta_0 < \theta_1 < 2\pi$ and $\theta_1 - \theta_0$ sufficiently small

$$\int_{\theta_0}^{\theta_1} \text{Re}\left\{ \alpha \left(1 + \frac{zw''(z)}{w'(z)} \right) - \alpha \frac{zw'(z)}{w(z)} + \frac{zw'(z)}{w(z)} \left[\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}+w}{e^{it}-w} du(t) \right] \right\} d\theta$$

$$= \int_{\theta_0}^{\theta_1} \text{Re}\left\{ \alpha \left[1 + \frac{zw''(z)}{w'(z)} \right] + (1-\alpha) \frac{zw'(z)}{w(z)} \right\} d\theta$$

$$+ \text{Re}\left\{ \int_{\theta_0}^{\theta_1} \frac{zw'(z)}{e^{it}-w} du(t) d\theta \right\} \quad (8)$$

$$= \int_{\theta_0}^{\theta_1} \left[\frac{\partial J_w(\theta)}{\partial \theta} \right] d\theta + \text{Re}\left\{ \frac{1}{\pi} \int_0^{2\pi} \left[\int_{\theta_0}^{\theta_1} \frac{zw'(z)}{e^{it}-w} d\theta \right] du(t) \right\}. \quad (9)$$

The interchanging of the order of integration follows from the continuity in θ and t of the integrand of the second term in (8). Since $z = re^{i\theta}$ it follows for a fixed r that $dz = ire^{i\theta} d\theta$. This fact and remembering that $w(z)$ is one-to-one allows us to rewrite (9) as

$$J_w(\theta_1) - J_w(\theta_0) - \frac{1}{2\pi} \int_0^{2\pi} 2 \operatorname{Im}\left\{ \int_{w_0}^{w_1} d\big[\log(e^{it} - w) \big] \right\} du(t)$$

$$= J_w(\theta_1) - J_w(\theta_0) - \frac{1}{2\pi} \int_0^{2\pi} \left[2 \arg\left(\frac{e^{it} - w_1}{e^{it} - w_0} \right) \right] du(t) \qquad (10)$$

$$\geqslant J_w(\theta_1) - J_w(\theta_0) - 2 \max_{|\zeta|=1} \arg \frac{\zeta - w_1}{\zeta - w_0}. \qquad (11)$$

It follows that (7) is nonnegative whenever

$$J_w(\theta_1) - J_w(\theta_0) \geqslant 2 \max_{|\zeta|=1} \arg \frac{\zeta - w_1}{\zeta - w_0}. \qquad (12)$$

We now prove that

$$\max_{|\zeta|=1} \arg \frac{\zeta - w_1}{\zeta - w_0} = \arg \frac{\zeta_0 - w_1}{\zeta_0 - w_0} \qquad (13)$$

where $\zeta_0 = e^{i\theta_0}$ is the point on the oricycle defined by (4). For an arbitrary point ζ on the unit circle, let $\beta = \arg[(\zeta - w_1)/(\zeta - w_0)]$. Because we are determining the max of $\arg[(\zeta - w_1)/(\zeta - w_0)]$ we need only consider the ζ's in the half plane determined by the line through w_0 and w_1 that makes β positive. We let w_ζ be the point of intersection of the oricycle C_0 and the line segment $\overline{\zeta w_0}$. Denote by γ the angle $w_1 w_\zeta w_0$. Then, since ζ is not inside C_0, it is clear that $\beta \leqslant \gamma$. From the simple geometric properties of a circle it follows that γ equals the angle $w_1 \zeta_0 w_0$. Hence (13) follows. Since the central angle $w_1 \eta_0 w_0$ equals twice the angle $w_1 \zeta_0 w_0$ condition (5) implies that $g(U)$ is α-starlike. Therefore $f(X)$ is α-starlike for each f in $S^*(\alpha)$.

To prove that (5) is a necessary condition for X to be in $H_{S^*(\alpha)}$ assume there exists an X in $H_{S^*(\alpha)}$ with the Riemann map w^* and that X contains two points $w_0^* = w^*(re^{i\theta_0})$, $w_1^* = w^*(re^{i\theta_1})$, $0 < \arg[w_1^*/w_0^*] \leqslant \pi$, $0 \leqslant \theta_1 < \theta_2 < 2\pi$ such that

$$J_{w^*}(\theta_1) - J_{w^*}(\theta_0) < \arg \frac{\eta_0^* - w_1^*}{\eta_0^* - w_0^*} = 2 \arg \frac{\zeta_0^* - w_1^*}{\zeta_0^* - w_0^*}$$

with ζ_0^*, η_0^* defined as functions of w_1^* and w_0^* as before. Let $\zeta_0^* = e^{i\theta_3}$. Choose $u = u_0(t)$ such that

$$u_0(t) = \begin{cases} 0, & 0 \leqslant t \leqslant \theta_3, \\ 2\pi, & \theta_3 < t \leqslant 2\pi \end{cases}$$

in

$$\alpha\left[1 + \frac{z f_{u_0}''(z)}{f_{u_0}'(z)} \right] + (1 - \alpha) \frac{z f_{u_0}'(z)}{f_{u_0}(z)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \, du(t). \qquad (14)$$

This defines f_{u_0} where f_{u_0} has the form

$$f_{u_0}(z) = \left[\frac{1}{\alpha} \int_0^z \zeta^{1/\alpha - 1}\left(1 - \bar{\zeta}_0^* \zeta\right)^{-2/\alpha} d\zeta \right]^{\alpha}.$$

This f_{u_0}, where the appropriate branches have been chosen in the above integral, is a rotation of the generalized Köebe function shown by many authors to be the extremal function for a number of problems over the class $S^*(\alpha)$. Replacing u by u_0 in (10) gives equality between lines (10) and (11). Thus, for the function g^* defined by $g^*(z) = f_{u_0}[w^*(z)]$ we obtain

$$J_{g^*}(\theta_1) - J_{g^*}(\theta_0) = J_{w^*}(\theta_1) - J_{w^*}(\theta_0) - 2 \arg \frac{\zeta_0^* - w_1^*}{\zeta_0^* - w_0^*} < 0.$$

It follows that $g^*(z)$ maps D onto X and that $b(g)$ is not in $S^*(\alpha)$ for any positive scalar b. This completes the proof of Theorem 2.

REMARK. When $\alpha = 1$ condition (5) in Theorem 2 becomes

$$\arg re^{i\theta_1}w'(re^{i\theta_1}) - \arg re^{i\theta_0}w'(re^{i\theta_0}) \geqslant \arg \frac{w(re^{i\theta_1}) - \eta_0}{w(re^{i\theta_0}) - \eta_0}. \qquad (15)$$

To see that (15) gives an analytic description of the geometric criterion of Pommerenke-Heins is a long but straightforward argument based essentially on the observation that equality in (15) exactly characterizes an oricycle.

One may ask if there is a geometric characterization of the sets in $H_{S^*(1)} = H_K$. The answer is that such sets are made up of arbitrary intersections of sets having oricycles as boundaries. Since oricycles appear in Theorem 2 for any α, one can ask for what α other than $\alpha = 1$, can an oricycle form the boundary of a set in $H_{S^*(\alpha)}$. It follows from Theorem 2 by a lengthy continuity argument that the only value of α for which a set in $H_{S^*(\alpha)}$ has an oricycle as its boundary is $\alpha = 1$. The author wishes to thank the referee for many helpful suggestions in this paper.

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