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ON ROBINSON'S $\frac{1}{2}$ CONJECTURE

ROGER W. BARNARD

ABSTRACT. In 1947, R. Robinson conjectured that if f is in S, i.e. a normalized univalent function on the unit disk, then the radius of univalence of $[zf(z)]'/2$ is at least $\frac{1}{2}$. He proved in that paper that it was at least .38. The conjecture has been shown to be true for most of the known subclasses of S. This author shows through use of the Grunski inequalities, that the minimum lower bound over the class S lies between .49 and .5.

Introduction. Let $\mathcal{Q}$ denote the class of analytic functions on the unit disk $U = \{z: |z| < 1\}$. Let S denote the univalent functions f in $\mathcal{Q}$ normalized by $f(0) = 1 - f'(0) = 0$. Denote by K, S^*, C, and Sp the standard subclasses of S consisting of functions that are convex starlike, close to convex and spirallike respectively. For a subclass X (possibly a singleton) of $\mathcal{Q}$ let $r_S(X)$ denote the minimum radius of univalence over all functions f in X. We use corresponding notation for the other subclasses of S. For example $r_{S^*}(X)$ denotes the minimum radius of starlikeness over all functions f in X.

For a function f in S define the operator Γ: $S \to \mathcal{Q}$ by $\Gamma f = (zf)\frac{1}{2}$. In 1947 R. Robinson [**10**] considered the problem of determining $r_S[\Gamma(S)]$. Robinson observed that for each f in S, $[\Gamma(f)]' \neq 0$ for $|z| < \frac{1}{2}$. He also noted that for the Koebe function k, $k(z) = z(1 - z)^{-2}$, $r_S(k) = r_{S^*}(k) = \frac{1}{2}$, which implies $r_S[\Gamma(S)] \leqslant \frac{1}{2}$. He in fact conjectured that $r_S[\Gamma(S)] = \frac{1}{2}$. He was able to show that $r_{S^*}[\Gamma(S)] > .38$.

There have been a number of papers (e.g. [**2**], [**3**], [**6**], [**7**], [**8**]) on the connection between the operator Γ and various subclasses of S. In these papers it has been shown that

$$r_K[\Gamma(K)] = r_{S^*}[\Gamma(S^*)] = r_C[\Gamma(C)] = r_{\mathrm{Sp}}[\Gamma(\mathrm{Sp})] = \tfrac{1}{2}$$

and that Γ preserves Rogosinski's class of typically real functions (not necessarily univalent) up to $|z| < \frac{1}{2}$. It was observed in [**2**] that with the exception of the result $r_{\mathrm{Sp}}[\Gamma(\mathrm{Sp})] = \frac{1}{2}$ these results follow directly from the S. Ruscheweyh-T. Sheil-Small theory [**11**]. They proved that, except for Sp, convolution by convex functions preserves the above subclasses of S. In order to obtain the related results in [**2**] one need only observe that for $f(z) = \Sigma a_n z^n$,

$$\Gamma[f(z)] = h * f(z) = \sum [(n + 1)/2] z^n * f(z) = \sum [(n + 1)/2] a_n z^n$$

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and that $h(z) = (z - z^2/2)(1-z)^{-2}$ is convex for $|z| < \frac{1}{2}$. As was shown in [2] most of the results that had been obtained on generalizations of the operator Γ on subclasses of S can also be obtained in a similar manner by the appropriate modifications of h. However, for the entire class S, if we let $r_0 = r_C(S) \approx .80$ from [5], it appears that the easily obtained lower bound for $r_S[\Gamma(S)]$ of $r_0/2 \approx .41$ is the most that can be obtained from the convolution operator method. It does show that $.41 < r_S[\Gamma(S)] \leqslant \frac{1}{2}$. In the present note the author, through use of the Grunsky inequalities, is able to prove that $.49 < r_S[\Gamma(S)] \leqslant \frac{1}{2}$.

Proof of Main Result. To find a lower bound for $r_S[\Gamma(S)]$ we consider the nonvanishing of

$$\frac{f(z) + zf'(z) - f(\zeta) - \zeta f'(\zeta)}{z - \zeta}.$$

By use of the minimum principle we may assume $|z| = |\zeta| < r$. Since f is in S we may divide through by $[f(z) - f(\zeta)]/(z - \zeta)$. Thus it suffices to find the largest r such that

$$1 + \frac{zf'(z) - \zeta f'(\zeta)}{f(z) - f(\zeta)} \neq 0, \qquad |z| = |\zeta| < r. \tag{1}$$

Consider for f in S the Grunsky coefficients defined by letting

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{n,m=0}^{\infty} d_{nm} z^n \zeta^m. \tag{2}$$

Putting $\zeta, z = 0$ respectively, in (2) we obtain

$$\log \frac{f(z)}{z} = \sum_{n=0}^{\infty} d_{n0} z^n, \qquad \log \frac{f(\zeta)}{\zeta} = \sum_{m=0}^{\infty} d_{0m} \zeta^m.$$

Hence

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = \log \frac{f(z)}{z} + \log \frac{f(\zeta)}{\zeta} + \sum_{n,m=1}^{\infty} d_{nm} z^n \zeta^m. \tag{3}$$

Although Grunsky's inequalities are usually stated in terms of the function F, on $|\xi| > 1$ defined by $F(\xi) = 1/f(1/\xi)$, it is more convenient for our purposes to express them directly in terms of f in S. To do this, we observe, by letting $z' = 1/z, \zeta' = 1/\zeta$, that

$$\begin{aligned}
&\log \frac{f(z) - f(\zeta)}{z - \zeta} - \log \frac{f(z)}{z} - \log \frac{f(\zeta)}{\zeta} \\
&\quad = \log \frac{f(1/z') - f(1/\zeta')}{f(1/z') f(1/\zeta') z'\zeta' [(1/z') - (1/\zeta')]} \\
&\quad = \log \frac{1/f(1/z') - 1/f(1/\zeta')}{z' - \zeta'} = \log \frac{F(z') - F(\zeta')}{z' - \zeta'} \\
&\quad = \sum_{n,m=1}^{\infty} d_{nm} (z')^{-n} (\zeta')^{-m} = \sum_{n,m=1}^{\infty} d_{nm} z^n \zeta^m.
\end{aligned}$$

Thus we can use the following form of Grunsky's inequalities (see Pommerenke [9, p. 60]). For f in S and d_{nm} defined by (2) we have for arbitrary complex x_n,

$$\sum_{n=1}^{\infty} n\left|\sum_{m=1}^{\infty} d_{nm}x_m\right|^2 \leqslant \sum_{n=1}^{\infty} \frac{|x_n|^2}{n} \tag{4}$$

provided the last series converges. Now, differentiating (3) with respect to z and ζ we see from the uniform convergence of the series in (3) for $|z| = |\zeta| \leqslant r < 1$ that

$$\frac{zf'(z)}{f(z)-f(\zeta)} - \frac{z}{z-\zeta} = \frac{zf'(z)}{f(z)} - 1 + \sum_{m,n=1}^{\infty} nd_{nm}z^n\zeta^m,$$

and

$$\frac{-\zeta f'(\zeta)}{f(z)-f(\zeta)} + \frac{\zeta}{z-\zeta} = \frac{\zeta f'(\zeta)}{f(\zeta)} - 1 + \sum_{n,m=1}^{\infty} md_{nm}z^n\zeta^m.$$

Adding these two expressions and rearranging we obtain

$$1 + \frac{zf'(z)-\zeta f'(\zeta)}{f(z)-f(\zeta)} = \frac{zf'(z)}{f(z)} + \frac{\zeta f'(\zeta)}{f(\zeta)} + \sum_{n,m=1}^{\infty} (n+m)d_{nm}z^n\zeta^m. \tag{5}$$

Thus from (1) we need to find the largest r for which the right-hand side of (5), which we denote by $T(z, \zeta)$, does not vanish for $|z| = |\zeta| < r$. We have by the use of Schwarz's inequality and (4) that

$$\begin{aligned}
\mathcal{R}\mathrm{e}\{T(z,\zeta)\} &\geqslant \mathcal{R}\mathrm{e}\left\{\frac{zf'(z)}{f(z)} + \frac{\zeta f'(\zeta)}{f(\zeta)}\right\} - \left|\sum_{n,m=1}^{\infty}(n+m)d_{nm}z^n\zeta^m\right| \\
&\geqslant 2\min_{|z|=r} \mathcal{R}\mathrm{e}\left\{\frac{zf'(z)}{f(z)}\right\} - \left(\left|\sum_{n=1}^{\infty}\sqrt{n}\,z^n\sum_{m=1}^{\infty}\sqrt{n}\,d_{nm}\zeta^m\right|\right. \\
&\qquad\qquad \left. + \left|\sum_{m=1}^{\infty}\sqrt{m}\,\zeta^m\sum_{n=1}^{\infty}\sqrt{m}\,d_{mn}z^n\right|\right) \\
&\geqslant 2\min_{|z|=r} \mathcal{R}\mathrm{e}\left\{\frac{zf'(z)}{f(z)}\right\} - \left(\sum_{n=1}^{\infty} nr^{2n}\right)^{1/2}\left[\sum_{n=1}^{\infty} n\left|\sum_{m=1}^{\infty} d_{nm}\zeta^m\right|^2\right]^{1/2} \\
&\qquad - \left(\sum_{m=1}^{\infty} mr^{2m}\right)^{1/2}\left[\sum_{m=1}^{\infty} m\left|\sum_{n=1}^{\infty} d_{mn}z^n\right|^2\right]^{1/2} \\
&\geqslant 2\min_{|z|=r} \mathcal{R}\mathrm{e}\left\{\frac{zf'(z)}{f(z)}\right\} - \left[\frac{r^2}{(1-r^2)^2}\right]^{1/2}\left(\sum_{n=1}^{\infty}\frac{r^{2n}}{n}\right)^{1/2}
\end{aligned}$$

$$-\left[\frac{r^2}{(1-r^2)^2}\right]^{1/2}\left(\sum_{m=1}^{\infty}\frac{r^{2m}}{m}\right)^{1/2}$$

$$\geqslant 2\min_{|z|=r}\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\}-2\left(\frac{r}{1-r^2}\right)\left(\log\frac{1}{1-r^2}\right)^{1/2}. \qquad (6)$$

Since f is in S we have the well-known inequality

$$\left|\log\frac{zf'(z)}{f(z)}\right|\leqslant\log\frac{1+r}{1-r},\ |z|\leqslant r,$$

where, in fact, for each real α, and z, $|z|=r<1$, there exists an f in S such that

$$\log\left[zf'(z)/f(z)\right]=e^{i\alpha}\log\left[(1+r)/(1-r)\right]$$

(see Jenkins [**4**, p. 110]). Thus, if we let $\log[zf'(z)/f(z)]=Re^{i\Phi}$, we can assume $R=\log[(1+r)/(1-r)]$. In order to find the minimum of (6) for all f in S we consider

$$\min_{f\in S}\ \min_{|z|=r}\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\}=\min_{\Phi}\ \mathrm{Re}\left\{\exp(Re^{i\Phi})\right\}$$

$$=\min_{\Phi}\left[\exp(R\cos\Phi)\right]\left[\cos(R\sin\Phi)\right].$$

Thus, from (6), we need to find the largest r for which

$$\min_{\Phi}\left[\exp(R\cos\Phi)\right]\left[\cos(R\sin\Phi)\right]\geqslant\frac{r}{1-r^2}\left[-\log(1-r^2)\right]^{1/2}. \qquad (7)$$

It is easy to see that the left-hand side of (7), call it LS, is a decreasing function of r while the right-hand side of (7), call it RS, is an increasing function of r. A computer checked calculation shows that for $r=.490$, RS $<.3379$ while LS $>.3393$ where the minimum value occurs when Φ is approximately 2.5 radians. We note that for $r=.491$, RS $>.3398$. Thus, inequality (1) holds for all f in S and $r\leqslant.49$. It follows that $r_S[\Gamma(S)]>.49$.

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Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506

Current address: Department of Mathematics, Texas Tech University, Lubbock, Texas 79409