ON THE PROXIMALITY OF COMPACT OPERATORS WITH RANGE IN $C(S)$

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ABSTRACT. It is proved that the space of compact operators on X into $C(S)$ is proximal in the corresponding space of bounded operators, if X is a Hilbert space, a space l_p , $1 < p < +\infty$, or the space c_0 .

Let X be a Banach space, E a closed subspace of X , $x \in X$. We denote by $P_E(x)$ the set of all elements of best approximation of x in E , i.e., $P_E(x) = \{e \in E; \|x - e\| = \inf_{y \in E} \|x - y\|\}$. E is said to be proximal, if $P_E(x) \neq \emptyset$ for every $x \in X$.

Recently, many authors have investigated the problem of best approximation of bounded linear operators $B(X, Y)$ on a Banach space X into a Banach space Y by elements of the corresponding space of compact operators $K(X, Y)$ (see, e.g., [2]–[5]). Although, in some special cases, a lot of information is known, the following problem still remains open, even for a number of standard Banach spaces: For which Banach spaces X and Y is $K(X, Y)$ proximal in $B(X, Y)$? The purpose of this paper is to give an affirmative answer to this question in the case that X is a Hilbert space, a space l_p , $1 < p < +\infty$, or the space c_0 , and Y is $C(S)$, the space of all continuous scalar functions on a compact Hausdorff space S .

To obtain one general proof valid for all mentioned cases we make the following definitions.

DEFINITION 1. A Banach space X is said to have the property (P_1) if there is a continuous strictly increasing function $\Psi: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ with $\Psi(0) = 0$ and $\Psi(1) = 1$ such that the following is true: For every bounded net $\{x_\alpha\}$ in the dual X^* of X w^* -converging to 0, every $y \in X^*$ and each $\varepsilon > 0$ there exists an α_0 such that

$$|\Psi(\|x_\alpha + y\|) - \Psi(\|x_\alpha\|) - \Psi(\|y\|)| < \varepsilon \quad (1)$$

for every $\alpha \geq \alpha_0$.

DEFINITION 2. A Banach space X is said to have the property (P_2) if for every bounded net $\{x_\alpha\}$ in X^* with $w^*\text{-}\lim x_\alpha = 0$ and $\lim \|x_\alpha\| = 1$, every $y \in X^*$ with $\|y\| \leq 1$ and every $\varepsilon > 0$ there is a cofinal subnet $\{x_\beta\}$ of $\{x_\alpha\}$ such that for every β

$$\|x_\beta - y\| \geq 1 - \varepsilon.$$

The following is obvious.

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PROPOSITION 1. *Every Banach space with the property (P_1) has the property (P_2) .*

Now we give some examples.

PROPOSITION 2. *Every Hilbert space H has the property (P_1) .*

PROOF. Putting $\Psi(t) = t^2$, $t \in \mathbb{R}^+$, for any net $\{x_\alpha\} \subset H$ w^* -converging to 0 and any $y \in H$, the claim follows from the relation

$$\Psi(\|x_\alpha + y\|) = \Psi(\|x_\alpha\|) + \Psi(\|y\|) + (x_\alpha, y) + (y, x_\alpha).$$

PROPOSITION 3. *Every space l_p , $1 < p < +\infty$, has the property (P_1) . The space c_0 has the property (P_1) .*

PROOF. For $1 \leq q < +\infty$ put $\Psi(t) = t^q$. Let $\varepsilon > 0$, $x_\alpha = \{\xi_i^\alpha\} \in l_q$ with w^* -lim $x_\alpha = 0$, and $y = \{\eta_i\} \in l_q$ be given. Put $M = \sup \|x_\alpha\|$. Since Ψ is uniformly continuous in $[0, M + 1]$, there is an $n_0 \in \mathbb{N}$ such that

$$\left| \sum_{i=n_0+1}^{\infty} |\xi_i^\alpha + \eta_i|^q - \sum_{i=n_0+1}^{\infty} |\xi_i^\alpha|^q \right| < \varepsilon/4$$

for every α , and such that

$$\sum_{i=n_0+1}^{\infty} |\eta_i|^q < \varepsilon/4.$$

Find an α_0 such that

$$| |\xi_i^\alpha + \eta_i|^q - |\eta_i|^q | < \varepsilon/4n_0$$

for $\alpha > \alpha_0$, $1 \leq i \leq n_0$, and

$$\sum_{i=1}^{n_0} |\xi_i^\alpha|^q < \varepsilon/4$$

for $\alpha > \alpha_0$. It follows that

$$\begin{aligned} & |\Psi(\|x_\alpha + y\|) - \Psi(\|x_\alpha\|) - \Psi(\|y\|)| \\ &= \left| \sum_{i=1}^{\infty} |\xi_i^\alpha + \eta_i|^q - \sum_{i=1}^{\infty} |\xi_i^\alpha|^q - \sum_{i=1}^{\infty} |\eta_i|^q \right| < \varepsilon. \end{aligned}$$

This completes the proof.

Let X be a Banach space. By the representation theorem VI.7.1 of [1], there is a one-to-one correspondence between $B(X, C(S))$ and w^* -continuous maps $u: S \rightarrow X^*$ such that for $L \in B(X, C(X))$

$$Lx(t) = u(t)x, \quad x \in X, t \in S,$$

and

$$\|L\| = \sup_{t \in S} \|u(t)\|.$$

The operator L is compact iff u is continuous with the norm topology in X^* .

In the next theorem a lower bound for the distance

$$\text{dist}(L, K(X, C(S))) = \inf_{T \in K(X, C(S))} \|L - T\|$$

of an arbitrary $L \in B(X, C(S))$ to $K(X, C(S))$ is given.

THEOREM 1. *Let $L \in B(X, C(S))$, where X has the property (P_2) . Let $u: S \rightarrow X^*$ be the corresponding w^* -continuous map. Then we have*

$$\text{dist}(L, K(X, C(S))) \geq \sup_{t \in S} \limsup_{s \rightarrow t} \|u(t) - u(s)\|.$$

PROOF. For $t \in S$ denote $d(t) = \limsup_{s \rightarrow t} \|u(t) - u(s)\|$. For an operator $T \in K(X, C(S))$ let $v: S \rightarrow X^*$ be the corresponding map. We have to show that for every $t \in S$

$$\|L - T\| = \sup_{s \in S} \|u(s) - v(s)\| \geq d(t),$$

or, in other words, that for every $t \in S$ and every $\varepsilon > 0$ there is an $s \in S$ such that

$$\|u(s) - v(s)\| \geq d(t) - \varepsilon.$$

Suppose that this is not true. Then there are a $t_0 \in S$ and an $\varepsilon_0 > 0$ such that for every $s \in S$ we have

$$\|u(s) - v(s)\| < d(t_0) - \varepsilon_0. \quad (2)$$

There is a net $\{s_\alpha\} \subset S$ with $\lim s_\alpha = t_0$ and

$$\lim \|u(s_\alpha) - u(t_0)\| = d(t_0). \quad (3)$$

Since v is continuous with the norm topology in X^* , there is an α_0 such that for every $\alpha \geq \alpha_0$ we have

$$\|v(s_\alpha) - v(t_0)\| < \varepsilon_0/2. \quad (4)$$

Consequently, by (2) and (4),

$$\begin{aligned} \|u(s_\alpha) - v(t_0)\| &\leq \|u(s_\alpha) - v(s_\alpha)\| + \|v(s_\alpha) - v(t_0)\| \\ &< d(t_0) - \varepsilon_0/2 \end{aligned} \quad (5)$$

for every $\alpha \geq \alpha_0$. Denoting for $s \in S$

$$u_1(s) = (1/d(t_0))u(s), \quad v_1(s) = (1/d(t_0))(v(s) - u(s)),$$

we obtain $w^*\text{-}\lim(u_1(s_\alpha) - u_1(t_0)) = 0$ and, by (3), $\lim \|u_1(s_\alpha) - u_1(t_0)\| = 1$. Further, by (5), we have for $\alpha \geq \alpha_0$

$$\begin{aligned} \|u_1(s_\alpha) - u_1(t_0) - v_1(t_0)\| &= (1/d(t_0))\|u(s_\alpha) - u(t_0) - v(t_0) + u(t_0)\| \\ &= (1/d(t_0))\|u(s_\alpha) - v(t_0)\| < 1 - \varepsilon_0/(2d(t_0)). \end{aligned}$$

Hence X cannot have the property (P_2) which contradicts the assumption.

Now we formulate our main result.

THEOREM 2. *If a Banach space X has the property (P_1) , then $K(X, C(S))$ is proximal in $B(X, C(S))$.*

PROOF. Let L be an arbitrary bounded operator in $B(X, C(S))$, $u: S \rightarrow X^*$ the corresponding w^* -continuous map. Denote $d(t) = \limsup_{s \rightarrow t} \|u(t) - u(s)\|$, $d = \sup_{t \in S} d(t)$. By Theorem 1 it is sufficient to show that there exists a function $v: S \rightarrow X^*$ which is continuous with the norm topology in X^* such that

$$\sup_{t \in S} \|v(t) - u(t)\| \leq d. \quad (6)$$

Since X has the property (P_1) , there is a strictly increasing continuous function $\Psi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\Psi(0) = 0$ and $\Psi(1) = 1$ satisfying (1). Define a set valued mapping $\Phi: S \rightarrow 2^{X^*}$ by

$$\Phi(t) = \{x \in X^*; \|x - u(t)\| \leq \Psi^{-1}(\Psi(d) - \Psi(d(t)))\}.$$

To accomplish the proof we need the following two lemmas.

LEMMA 1. *For every $t \in S$ and every $\varepsilon > 0$ there is a neighborhood U of t such that for every $s \in U$ we have*

$$\Psi(d(t)) \geq \Psi(d(s)) + \Psi(\|u(t) - u(s)\|) - \varepsilon.$$

PROOF. Assume that there is a $t \in S$ and an $\varepsilon_0 > 0$ such that every neighborhood of t contains a point s with

$$\Psi(d(t)) < \Psi(d(s)) + \Psi(\|u(t) - u(s)\|) - \varepsilon_0. \quad (7)$$

There is a neighborhood U_0 of t such that for every $s \in U_0$ we have

$$\Psi(\|u(t) - u(s)\|) \leq \Psi(d(t)) + \varepsilon_0/6. \quad (8)$$

Find an $s_0 \in U_0$ for which (7) holds. Further, there is an $s_1 \in U_0$ such that

$$\Psi(\|u(s_0) - u(s_1)\|) \geq \Psi(d(s_0)) - \varepsilon_0/6, \quad (9)$$

and, since u is w^* -continuous, such that

$$\begin{aligned} & \Psi(\|u(t) - u(s_0) + u(s_0) - u(s_1)\|) \\ & \geq \Psi(\|u(t) - u(s_0)\|) + \Psi(\|u(s_0) - u(s_1)\|) - \varepsilon_0/6. \end{aligned} \quad (10)$$

Consequently, by (7)–(10),

$$\begin{aligned} & \Psi(d(s_0)) + \Psi(\|u(t) - u(s_0)\|) - \varepsilon_0 > \Psi(d(t)) \\ & \geq \Psi(\|u(t) - u(s_1)\|) - \varepsilon_0/6 \\ & \geq \Psi(\|u(t) - u(s_0)\|) + \Psi(\|u(s_0) - u(s_1)\|) - 2\varepsilon_0/6 \\ & \geq \Psi(\|u(t) - u(s_0)\|) + \Psi(d(s_0)) - \varepsilon_0/2. \end{aligned}$$

It follows that $-\varepsilon_0/2 > 0$. A contradiction.

LEMMA 2. *For every $t \in S$ and every $\varepsilon > 0$ there is a neighborhood U of t such that for every $s \in U$ and every $x \in X^*$*

$$\|x - u(t)\| \leq \Psi^{-1}(\Psi(d) - \Psi(d(t)))$$

implies

$$\|x - u(s)\| \leq \Psi^{-1}(\Psi(d) - \Psi(d(s))) + \varepsilon.$$

PROOF. Given $\varepsilon > 0$ there is, since Ψ^{-1} is uniformly continuous in $[0, \Psi(d) + 1]$, a $\delta > 0$ such that for every $t_1, t_2 \in [0, \Psi(d) + 1]$ $t_1 \leq t_2 + \delta$ implies

$$\Psi^{-1}(t_1) \leq \Psi^{-1}(t_2) + \varepsilon. \quad (11)$$

By Lemma 1 there is a neighborhood U_1 of t such that for every $s \in U_1$ we have

$$\Psi(d(t)) \geq \Psi(d(s)) + \Psi(\|u(t) - u(s)\|) - \delta/2. \quad (12)$$

Further, since u is w^* -continuous, there is a neighborhood U_2 of t such that

$$\begin{aligned} \Psi(\|x - u(t) + u(t) - u(s)\|) \\ \leq \Psi(\|x - u(t)\|) + \Psi(\|u(t) - u(s)\|) + \delta/2 \end{aligned} \quad (13)$$

for every $s \in U_2$. Then, by (12) and (13), for each $s \in U = U_1 \cap U_2$

$$\begin{aligned} \Psi(\|x - u(s)\|) &= \Psi(\|x - u(t) + u(t) - u(s)\|) \\ &\leq \Psi(\|x - u(t)\|) + \Psi(\|u(t) - u(s)\|) + \delta/2 \\ &\leq \Psi(d) - \Psi(d(t)) + \Psi(\|u(t) - u(s)\|) + \delta/2 \\ &\leq \Psi(d) - \Psi(d(s)) + \delta. \end{aligned}$$

Consequently, by (11),

$$\|x - u(s)\| \leq \Psi^{-1}(\Psi(d) - \Psi(d(s))) + \varepsilon, \quad s \in U,$$

which completes the proof of Lemma 2.

Now, we can accomplish the proof of Theorem 2. It follows from Lemma 2 that Φ is lower semicontinuous (cf. [7, Proposition 2.1]). Hence, by E. Michael's selection Theorem 3.2" of [7], Φ has a continuous selection v which obviously fulfils (6). This completes the proof of Theorem 2.

COROLLARY 3. *If X is a Hilbert space, a space l_p , $1 < p < +\infty$, or the space c_0 , then $K(X, C(S))$ is proximal in $B(X, C(S))$.*

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