A BOUND FOR DECOMPOSITIONS OF m -CONVEX SETS WHOSE LNC POINTS LIE IN A HYPERPLANE¹

MARILYN BREEN

ABSTRACT. A set S in R^d is said to be m -convex, $m \geq 2$, if and only if for every m points in S , at least one of the line segments determined by these points lies in S . Let S denote a closed m -convex set in R^d , and assume that the set of lnc points of S lies in a hyperplane. Then S is a union of $f(m)$ or fewer convex sets, where f is defined inductively as follows: $f(2) = 1$, $f(3) = 2$, and $f(m) = f(m-2) + 3$ for $m \geq 4$. Moreover, for $d \geq 3$, an example reveals that the best bound is no lower than $g(m)$, where $g(m) = f(m)$ for $2 < m < 5$ and for $m = 7$, and $g(m) = g(m-3) + 4$ otherwise.

1. Introduction. Let S be a subset of R^d . The set S is said to be m -convex, $m \geq 2$, if and only if for every m distinct points in S , at least one of the line segments determined by these points lies in S . A point x in S is said to be a *point of local convexity* of S if and only if there is some neighborhood N of x such that $N \cap S$ is convex. If S fails to be locally convex at some point q in S , then q is called a *point of local nonconvexity* (lnc point) of S . As in [1], the following terminology will be used: For x, y in S , we say x *sees* y *via* S if and only if the corresponding segment $[x, y]$ lies in S . Points $x_1, \dots, x_n$ in S are *visually independent via* S if and only if for $1 \leq i < j \leq n$, x_i does not see x_j via S . For x in S , the set of all points of S which see x via S is called the *star of x in S* , denoted S_x . Throughout the paper, $\text{cl } S$, $\text{int } S$, and $\text{rel int } S$ will be used to denote the closure, interior, and relative interior, respectively, of the set S .

Several decomposition theorems have been obtained for closed m -convex sets S in the plane [4], [3], [2], and if we further require the set of lnc points of S to lie on a line, then S will be a union of $m-1$ or fewer convex sets [1]. Here we investigate an analogous problem in R^d , $d \geq 3$, obtaining a decomposition theorem for closed m -convex sets in R^d whose lnc points lie in a hyperplane. Moreover, an example reveals that for $d \geq 3$ and $m \geq 4$, the planar bound $m-1$ is no longer possible.

2. The results.

THEOREM 1. *Let $S = \text{cl}(\text{int } S)$ be an m -convex set in R^d , $m \geq 2$, and assume that the set Q of lnc points of S lies in a hyperplane H . Then S is a union of $f(m)$ or fewer convex sets, where the function f is defined inductively as follows:*

Received by the editors December 5, 1977.

AMS (MOS) subject classifications (1970). Primary 52A20, 52A40.

¹This research was sponsored in part by an Arts and Sciences Summer Fellowship at the University of Oklahoma.

$f(2) = 1$, $f(3) = 2$, and $f(m) = f(m - 2) + 3$ for $m \geq 4$.

PROOF. We use an inductive argument. The result is trivial for $m = 2$, and if $m = 3$, it follows immediately from [1, Theorem 1]. Hence suppose that $m \geq 4$ and assume that the theorem is true for all integers k , $2 \leq k < m$, to prove for m .

Let H_1 and H_2 denote distinct open halfspaces determined by the hyperplane H . By the m -convexity of S , it is easy to see that for $i = 1, 2$, each set $S \cap H_i$ has at most $m - 1$ components, each starshaped. Furthermore, standard arguments involving a result by Valentine [5, Corollary 1] reveal that each of these components is convex. Since $S = \text{cl}(\text{int } S)$, every point of S must lie in the closure of at least one of these components.

We show that, without loss of generality, we may assume every point of $S \cap H$ lies in $\text{cl}(S \cap H_1) \cap \text{cl}(S \cap H_2)$: Otherwise, for some point x in $S \cap H$ and for an appropriate labeling of the halfspaces H_1 and H_2 , $x \notin \text{cl}(S \cap H_1)$. Then since $S = \text{cl}(\text{int } S)$, $x \in \text{cl}(S \cap H_2)$. By previous comments, for some k , $1 \leq k \leq m - 1$, x is in the closure of exactly k components of $S \cap H_2$, and if we choose one point from each of these components, clearly these points will be visually independent via S . If $k = m - 1$, then it is easy to show that $S_x = S$, $S = \text{cl}(S \cap H_2)$, and S is a union of $m - 1 < f(m)$ or fewer convex sets. If $k \leq m - 2$, then a proof similar to [1, Lemma 1] may be used to show that the set $S' \equiv \text{cl}(S \sim S_x)$ is at most $(m - k)$ -convex, $S' = \text{cl}(\text{int } S')$, and the lnc points of S' are in Q . Hence by our induction hypothesis, S' is a union of $f(m - k)$ or fewer convex sets. It is not hard to show that $\text{cl}(\text{int } S_x)$ is a union of k convex sets, and we conclude that

$$S = \text{cl}(\text{int } S_x) \cup \text{cl}(S \sim S_x)$$

is a union of $f(m - k) + k \leq f(m)$ or fewer convex sets, the desired result. Hence throughout the remainder of the proof we may assume that

$$S \cap H \subseteq \text{cl}(S \cap H_1) \cap \text{cl}(S \cap H_2).$$

Similarly, we may assume that for every component C of $S \sim H$, $\text{cl } C \cap H$ is a full $(d - 1)$ -dimensional: Otherwise, for some component C of $S \sim H$, S will be the union of the convex set C , and the $(m - 1)$ -convex set $\text{cl}(S \sim C)$, giving a decomposition of S into $1 + f(m - 1) \leq f(m)$ or fewer convex sets, again the appropriate bound.

To complete the proof, we consider two cases:

Case 1. Assume for the moment that there exist some lnc point q of S and some component C of $S \sim H$ such that $q \in \text{rel int}(\text{cl } C \cap H)$. (Recall that $\text{cl } C \cap H$ is $(d - 1)$ -dimensional.) For convenience of notation, let $C \subseteq H_1$. Then q is in the closure of k components of $S \cap H_2$. Moreover, since we are assuming that

$$S \cap H \subseteq \text{cl}(S \cap H_1) \cap \text{cl}(S \cap H_2),$$

it is clear that $2 \leq k$. Hence $2 \leq k \leq m - 1$. It is easy to see that the set

$\text{cl}(\text{int } S_q)$ is a union of $k + 1$ convex sets, each the closure of a component of $S \sim H$. If $k = m - 1$, then standard arguments show that $S = S_q$, and S is a union of $k + 1 = m \leq f(m)$ or fewer convex sets. If $k \leq m - 2$, then again the proof of [1, Lemma 1] reveals that the set $\text{cl}(S \sim S_q)$ is at most $(m - k)$ -convex and satisfies our induction hypothesis. Thus $\text{cl}(S \sim S_q)$ is a union of $f(m - k)$ or fewer convex sets, and S is a union of $f(m - k) + k + 1$ convex sets. But for $2 \leq k \leq m - 2$,

$$f(m - k) + k - 2 \leq f(m - k + k - 2) = f(m - 2),$$

so $f(m - k) + k + 1 \leq f(m - 2) + 3 = f(m)$, and S is a union of $f(m)$ or fewer convex sets, finishing the argument for Case 1.

Case 2. If Case 1 does not occur, then select any component C of $S \cap H_1$. By previous assumptions, there exists a component D of $S \cap H_2$ such that

$$\text{rel int}(\text{cl } C \cap H) \cap \text{rel int}(\text{cl } D \cap H) \neq \emptyset.$$

Moreover, D must be unique: Otherwise, for appropriate components D_1 and D_2 of $S \cap H_2$, $\text{cl } D_1 \cap \text{cl } D_2 \cap \text{rel int}(\text{cl } C \cap H) \neq \emptyset$, and there would be some lnc point of S in $\text{rel int}(\text{cl } C \cap H)$, contradicting our hypothesis for Case 2. Similarly, C is the unique component of $S \cap H_1$ for which $\text{rel int}(\text{cl } C \cap H) \cap \text{rel int}(\text{cl } D \cap H) \neq \emptyset$, and $\text{cl } C \cap H = \text{cl } D \cap H$. Note that the set $\text{cl } C \cup \text{cl } D$ is j -convex, where j is either 2 or 3. Now since $S = \text{cl}(\text{int } S)$ and $\text{cl } C \cap H = \text{cl } D \cap H$, it is easy to show that for any point z in $\text{cl } C \cup \text{cl } D$ and for any w in $S \sim (\text{cl } C \cup \text{cl } D)$, there is a sequence $\{w_n\}$ in S converging to w such that no w_n sees z via S . Hence standard arguments similar to those in [1, Lemma 1] may be used to show that the set $\text{cl}(S \sim (\text{cl } C \cup \text{cl } D))$ is $(m - j + 1)$ -convex and satisfies our induction hypothesis. We conclude that S is a union of $(j - 1) + f(m - j + 1) \leq f(m)$ or fewer convex sets. This completes Case 2 and finishes the proof of the theorem.

It is interesting to note that for $d \geq 3$, without the requirement that $S = \text{cl}(\text{int } S)$, the theorem fails and, in fact, no finite decomposition is possible. (See [1, Remarks 1 and 2].)

In conclusion, the following example reveals that for $d \geq 3$, the best bound is no lower than $g(m)$, where $g(m) = f(m)$ for $2 \leq m \leq 5$ and for $m = 7$, and $g(m) = g(m - 3) + 4$ otherwise.

EXAMPLE 1. Let g be a function defined inductively as follows: $g(2) = 1$, $g(3) = 2$, $g(4) = 4$, and $g(m) = g(m - 3) + 4$ for $m \geq 5$. Then for each $m \geq 2$ and $d \geq 3$, there corresponds an m -convex set $T_{m,d}$ in R^d which satisfies the hypothesis of Theorem 1 and which is expressible as a union of $g(m)$ and no fewer convex sets.

Examples for $m = 2$ and $m = 3$ are trivial. Hence let $m = 4$ and, for convenience of notation, let $d = 3$. Define $x_1 = (2, 0, 0)$, $x_2 = (0, 2, 0)$, $x_3 = (-2, 0, 0)$, $x_4 = (0, -2, 0)$. Define $R_i \equiv \text{conv}\{x_j : j \neq i\}$, $1 \leq i \leq 4$, and let $y_1 = (-1, 0, 1)$, $y_3 = (1, 0, 1)$, $y_2 = (0, -1, -1)$, $y_4 = (0, 1, -1)$. (Note that,

looking down on the x - y plane from the positive z axis, y_i is above the corresponding R_i set for $i = 1, 3$, below for $i = 2, 4$.) Finally, let

$$T_{4,3} = \bigcup \{ \text{conv}(R_i \cup \{y_i\}) : 1 \leq i \leq 4 \}.$$

Then $T_{4,3}$ is a 4-convex set which is a union of 4 and no fewer convex sets. This example may be generalized to higher dimensions to construct $T_{4,d}$.

Inductively, for $m \geq 5$, let $T_{m,d}$ be the disjoint union of $T_{m-3,d}$ and $T_{4,d}$. Then $T_{m,d}$ will be an m -convex set in R^d expressible as a union of $g(m-3) + 4 = g(m)$ and no fewer convex sets. Hence the planar bound of $m-1$ fails when $d \geq 3$ and $m \geq 4$.

REFERENCES

1. Marilyn Breen, *M-convex sets whose lnc points lie in a hyperplane*, J. London Math. Soc. **16** (1977), 529-535.
2. Marilyn Breen and David C. Kay, *General decomposition theorems for m -convex sets in the plane*, Israel J. Math. **24** (1976), 217-233.
3. H. G. Eggleston, *A condition for a compact plane set to be a union of finitely many convex sets*, Proc. Cambridge Philos. Soc. **76** (1974), 61-66.
4. F. A. Valentine, *A three point convexity property*, Pacific J. Math. **7** (1957), 1227-1235.
5. ———, *Local convexity and L_n sets*, Proc. Amer. Math. Soc. **16** (1965), 1305-1310.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OKLAHOMA, NORMAN, OKLAHOMA 73019