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A THEOREM OF BEURLING AND TSUJI IS BEST POSSIBLE

SHINJI YAMASHITA

ABSTRACT. We shall show that Beurling-Tsuji's theorem (see Theorem A) is, in a sense, best possible. For each pair a, $b \in (0, +\infty)$ there exists a function f holomorphic in $\{|z| < 1\}$ such that the Euclidean area of the Riemannian image of each non-Euclidean disk of non-Euclidean radius a, is bounded by b, and such that f has finite angular limit nowhere on the unit circle.

1. Introduction. Let $D = \{|z| < 1\}$, and let $\Gamma = \{|z| = 1\}$. For a function f holomorphic in D, and for a subset E of D we use the notation

$$A(E, f) = \iint_E |f'(z)|^2 \, dx \, dy, \qquad z = x + iy.$$

The following theorem is due to A. Beurling and M. Tsuji.

THEOREM A ([1], [4], see [5, Theorem VIII.49, p. 344]). *Let f be a function holomorphic in D with $A(D, f) < +\infty$. Then f has a finite angular limit at each point of Γ except for a set of zero logarithmic capacity.*

We shall show that extensions of Theorem A are, in a sense, impossible. Consider the non-Euclidean hyperbolic metric

$$\sigma(w, z) = \frac{1}{2} \log \frac{|1 - \bar{z}w| + |z - w|}{|1 - \bar{z}w| - |z - w|}, \qquad z, w \in D,$$

to define

$$H(z, a) = \{w \in D; \sigma(w, z) < a\}, \qquad z \in D, a \in (0, +\infty].$$

We let $F(a, b)$ be the family of all holomorphic functions f in D such that, for each $z \in D$, $A(H(z, a), f) \leqslant b$, where $a \in (0, +\infty]$ and $b \in (0, +\infty)$. Then, f of Theorem A belongs to $F(+\infty, b)$ with $b = A(D, f)$.

THEOREM 1. *Given $a \in (0, +\infty)$ and $b \in (0, \infty)$, we may find $f \in F(a, b)$ such that neither* $\operatorname{Re} f$ *nor* $\operatorname{Im} f$ *has a finite angular limit at any point of* Γ.

Thus, f has not a finite angular limit at any point of Γ.

2. Bloch function. A function f in D is called Bloch [3] if f is holomorphic in D and if

$$\beta(f) = \sup_{z \in D} (1 - |z|^2)|f'(z)| < +\infty.$$

Let $B(c)$ be the family of all Bloch functions f with $\beta(f) \leqslant c$, $c \in (0, +\infty)$.

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THEOREM 2. (2.1) *If* $a \in (0, +\infty)$, *then each* $f \in B(c)$ *is a member of* $F(a, b)$ *with*

$$b = \pi c^2 p(a)^2 / \left(1 - p(a)^2\right),$$

where

$$p(a) = (e^{2a} - 1)/(e^{2a} + 1).$$

(2.2) *If* $a \in (0, +\infty]$, $b \in (0, +\infty)$, *and if* $f \in F(a, b)$, *then* $f \in B(c)$ *with*

$$c^2 = b/\left[\pi p(a)^2\right] \qquad (p(+\infty) = 1).$$

PROOF. (2.1) Since

$$|f'(w)| \leqslant c(1 - |w|^2)^{-1}, \qquad w \in D,$$

it follows that, for each $z \in D$,

$$A(H(z, a), f) \leqslant c^2 \iint_{H(z,a)} (1 - |w|^2)^{-2} \, dx \, dy$$

$$= c^2 \iint_{|w| < p(a)} (1 - |w|^2)^{-2} dx \, dy = b \quad (w = x + iy),$$

because $(1 - |w|^2)^{-2} dx\, dy$ is invariant under non-Euclidean transformations.

(2.2) Set $p = p(a)$, and for each fixed $z \in D$, set

$$g(w) = f\left((pw + z)/(1 + \bar{z}pw)\right), \qquad w \in D.$$

Then

$$\pi p^2 (1 - |z|^2)^2 |f'(z)|^2 = \pi |g'(0)|^2 \leqslant A(D, g) = A\left(H(z, a), f\right) \leqslant b.$$

Therefore, $\beta(f) \leqslant c$, whence $f \in B(c)$.

3. Proof of Theorem 1. We shall make use of the two lemmata due to P. A. Lappan:

LEMMA 1 [**2**, p. 113]. *There exists a holomorphic function g in D, satisfying*

$$\sup_{z \in D} (1 - |z|) |g(z)| \leqslant 2, \tag{3.1}$$

and

$$\limsup_{0 < r \to 1-} (1 - r) |g(r\zeta)| > 0 \tag{3.2}$$

for each $\zeta \in \Gamma$.

LEMMA 2 [**2**, Lemma 3]. *Let f be a holomorphic function in D such that*

$$\limsup_{0 < r \to 1-} (1 - r) |f'(r\zeta)| > 0 \tag{3.3}$$

at a point $\zeta \in \Gamma$. *Then, neither* $\operatorname{Re} f$ *nor* $\operatorname{Im} f$ *has a finite angular limit at* ζ.

We note that our Lemma 2 is worded differently than Lemma 3 of [**2**], but the content is equivalent.

For the proof of Theorem 1 we choose $k > 0$ such that

$$k^2 = 16\pi p(a)^2/\left[b\left(1 - p(a)^2\right)\right]. \tag{3.4}$$

Let f be a function holomorphic in D such that $f' = k^{-1}g$, where g is the function in Lemma 1. Then it follows from (3.1) that $f \in B(c)$ with $c = 4k^{-1}$. It follows from Theorem 2, (2.1), together with (3.4) that $f \in F(a, b)$. It further follows from (3.2) that (3.3) is true at every point ζ of Γ. Thus, our assertion on the angular limits of $\operatorname{Re} f$ and $\operatorname{Im} f$ follows from Lemma 2.

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Department of Mathematics, Tokyo Metropolitan University, Fukazawa, Setagaya-ku, Tokyo 158, Japan