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EVERY L_p OPERATOR IS AN L_2 OPERATOR

W. B. JOHNSON[1] AND L. JONES

ABSTRACT. If T is a bounded linear operator on $L_p(\mu)$, $1 < p < \infty$, μ a probability measure, then, after an appropriate change of density, T acts as a bounded operator on L_2.

A *change of density* is a strictly positive element, g, of $L_1(\mu)$ which satisfies $\int g\, d\mu = 1$. Such a g induces an isometry $\varphi = \varphi(g, p)$ from $L_p(d\mu)$ onto $L_p(g d\mu)$ in an obvious way; namely,

$$\varphi f = g^{-1/p} f.$$

The theorem mentioned in the abstract can now be stated formally.

THEOREM 1. *Let $1 \leqslant p < \infty$ and suppose that T is a bounded linear operator on $L_p(d\mu)$ for some probability measure μ. Then there is change of density g so that $\varphi T \varphi^{-1}$ is a bounded linear operator on $L_2(g d\mu)$, where $\varphi = \varphi(g, p)$ is the isometry from $L_p(d\mu)$ onto $L_p(g d\mu)$ induced by g. That is, there is constant K and a change of density g such that*

$$\int \left| g^{-1/p} T\left(f g^{1/p}\right)\right|^2 g\, d\mu \leqslant K^2 \|T\|^2 \int |f|^2 g\, d\mu$$

for all $f \in L_p(g d\mu) \cap L_2(g d\mu)$. The constant $K = K(p)$ is independent of the operator T.

In the case $p = 1$, a stronger version of Theorem 1 can be easily proved:

PROPOSITION 1. *Suppose that T is a bounded linear operator on $L_1(d\mu)$, μ a probability measure, and $a > 1$. Then there is a change of density g so that $\|\varphi T \varphi^{-1}\|_{L_\infty(g d\mu)}$, the norm of $\varphi T \varphi^{-1}$ (where $\varphi = \varphi(g, 1)$) as an operator on $L_\infty(g d\mu)$, is less than $a \|T\|_{L_1(d\mu)}$.*

PROOF OF PROPOSITION 1. (This is a simplification by H. P. Rosenthal of the authors' original proof.) Assume, without loss of generality, that $\|T\|_{L_1(d\mu)} = 1$. Observe that if g is a change of density, and $\varphi = \varphi(g, 1)$, then $\|\varphi T \varphi^{-1}\|_{L_\infty(g d\mu)} \leqslant a$ if and only if T maps the order interval $[-g, g] \equiv \{f \in L_1(g d\mu)\colon |f| \leqslant |g|\ \mu\text{-a.e.}\}$ into the order interval $[-ag, ag]$.

Let $|T|$ be the modulus of T. Recall that $|T|$ is a positive linear contraction on $L_1(d\mu)$ and

$$|Tf| \leqslant |T|\,|f| \quad \text{for all } f \in L_1(d\mu).$$

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Let

$$\tilde{g} = 1 + \sum_{n=1}^{\infty} a^{-n}|T|^n 1,$$

and set

$$g = \left(\int \tilde{g}\, d\mu\right)^{-1} \tilde{g}.$$

(Note that $\tilde{g} \in L_1(d\mu)$ because $\| \, |T| \, \| = 1$ and $a > 1$.)

Suppose that $0 \leqslant f \leqslant \tilde{g}, f \in L_1(gd\mu)$.

Then

$$|T|f \leqslant |T|\tilde{g} \quad (\text{since } |T| \text{ is positive})$$
$$= \sum_{n=1}^{\infty} a^{-n+1}|T|^n 1 \leqslant a\tilde{g}.$$

Hence $|T|\,[0, g] \subseteq [0, ag]$, whence $T[-g, g] \subseteq [-ag, ag]$. □

To complete the proof of Theorem 1, it is enough, by duality, to consider the case $2 < p < \infty$. A stronger result, which is just a fixed point version of a theorem of Maurey's [3], is true.

PROPOSITION 2. *Suppose that T is a bounded linear operator from a subspace Y of $L_p(d\mu)$ into $L_p(d\mu)$, where μ is a probability measure and $2 < p < \infty$. Then there is a change of density $g \geqslant 1/2$ such that $\varphi g \varphi^{-1}$ (where $\varphi = \varphi(g, p)$) is bounded as an operator from $(\varphi Y, \|\cdot\|_{L_2(gd\mu)})$ into $L_2(gd\mu)$. In fact, $\|\varphi T \varphi^{-1}_{|\varphi Y}\|_{L_2(gd\mu)} \leqslant K\|T\|_{L_p(d\mu)}$ for some constant $K = K_p$ independent of the subspace Y and the operator T.*

PROOF. Maurey [3] proved that every operator from a subspace Y of $L_p(d\mu)$ into a Hilbert space induces a bounded linear operator from $(\varphi Y, \|\cdot\|_{L_2(gd\mu)})$ into the Hilbert space, for some change of density g, and where $\varphi = \varphi(g, p)$. If we restate Maurey's theorem for operators of the form

$$Y \xrightarrow{T} L_p(d\mu) \xrightarrow{\psi} L_p(hd\mu) \xrightarrow{i} L_2(hd\mu),$$

where h is a change of density, ψ is the isometry $\varphi(h, p)$, and i is the canonical injection, we obtain:

There is a constant $\lambda = \lambda_p$ so that, given any operator T from any subspace Y of $L_p(d\mu)$ into $L_p(d\mu)$ and any change of density h, there is a change of density g which satisfies, for every $f \in Y$,

$$\int |Tf|^2 h^{(p-2)/p}\, d\mu \leqslant \lambda^2 \int |f|^2 g^{(p-2)/p}\, d\mu.$$

(Note that

$$\int |f|^2 g^{(p-2)/p}\, d\mu = \int |g^{-1/p} f|^2 g\, d\mu = \|\varphi f\|^2_{L_2(gd\mu)}$$

where $\varphi = \varphi(g, p)$.)

Proposition 2 is just a fixed point version of this form of Maurey's theorem. We show that, given an operator T from a subspace Y of $L_p(d\mu)$ into $L_p(d\mu)$, there is a change of density $g \geqslant 1/2$ so that for every $f \in Y$,

$$\int |Tf|^2 g^{(p-2)/p}\, d\mu \leqslant 2\lambda^2 \int |f|^2 g^{(p-2)/p}\, d\mu. \tag{1}$$

This is, of course, almost trivial: Let $g_0 = 1$, and, using Maurey's theorem inductively on n, choose a change of density g_{n+1} so that, for $n = 0, 1, \ldots,$

$$\int |Tf|^2 g_{n+1}^{(p-2)/p}\, d\mu \leqslant \lambda^2 \int |f|^2 g_n^{(p-2)/p}\, d\mu$$

for every $f \in Y$.

Now define $\tilde{g}$ by

$$\tilde{g}^{(p-2)/p} = \sum_{n=0}^{\infty} 2^{-n-1} g_n^{(p-2)/p}, \tag{2}$$

and set

$$g = \left(\int \tilde{g}\, d\mu\right)^{-1} \tilde{g}.$$

Note that the series in (2) is absolutely convergent in $L_{p/(p-2)}(d\mu)$ to a function whose norm is at most one, so that $g \geqslant 1/2$.

Finally, if $f \in Y$, then

$$\int |Tf|^2 \tilde{g}^{(p-2)/p}\, d\mu = \sum_{n=0}^{\infty} 2^{-n-1} \int |Tf|^2 g_n^{(p-2)/p}\, d\mu$$
$$\leqslant \sum_{n=0}^{\infty} 2^{-n-1} \lambda^2 \int |f|^2 g_{n+1}^{(p-2)/p}\, d\mu \leqslant 2\lambda^2 \int |f|^2 \tilde{g}^{(p-2)/p}\, d\mu,$$

which gives (1). □

REMARKS. (1) Proposition 2 is used in **[2]** to show that a complemented subspace of Rosenthal's space X_p **[4]** which contains a complemented copy of X_p is isomorphic to X_p. In fact, it was the possibility of such an application which led to this note, and the condition "$g \geqslant 1/2$" in Proposition 2 appears because it is used in **[2]**.

(2) Theorem 1 has been generalized to the appropriate Banach lattice setting; see §5 of **[1]** for this generalization and an application.

(3) Results of the type proved here can be derived for general measures μ in a purely formal way from the results as we have stated them. We leave the formulations of the generalizations to the interested reader.

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DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, COLUMBUS, OHIO 43210