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AN EXTREMAL PROBLEM FOR FUNCTIONS OF POSITIVE REAL PART WITH APPLICATION TO A RADIUS OF CONVEXITY PROBLEM

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ABSTRACT. The functional $\operatorname{Re}\{zp'(z)/(p(z)+\beta+it)\}$, $\beta > -1$, $|z| \leqslant r$, $0 < r < 1$, is minimized for all real t over the class of functions of positive real part. The result is applied to obtain the radius of convexity for a family of regular functions.

1. Introduction. Let $S^*(\sigma)$, $\sigma < 1$, be the class of functions $g(z)$ regular in $\Delta = \{z: |z| < 1\}$ such that $g(0) = 0$, $g'(0) = 1$ and $\operatorname{Re}\{zg'(z)/g(z)\} > \sigma$ for all $z \in \Delta$.

We define the class $K(\sigma, \lambda)$, $\sigma < 1$, $\lambda < 1$, to be the set of functions $f(z)$ with $f(0) = 0, f'(0) = 1$, for which there exist $g(z) \in S^*(\sigma)$ and a real number α, $|\alpha| < \pi/2$, such that

$$\operatorname{Re}\left\{e^{i\alpha}\left[zf'(z)/g(z) - \lambda\right]\right\} > 0 \quad \text{for } z \in \Delta.$$

The radius of convexity of $K(\sigma, \lambda)$ is the greatest value of r, $0 < r \leqslant 1$, for which $\operatorname{Re}\{1 + zf''(z)/f(z)\} > 0$ for $|z| < r$ and for all $f(z)$ in $K(\sigma, \lambda)$. Jablonski and Wesolowski [1] found a lower bound for the radius of convexity of $K(0, \lambda)$ but the result is not sharp. Jankovics [2] obtained the radius of starlikeness of the class $\{f(z): \operatorname{Re}\{e^{i\alpha}[f(z)/z - \lambda]\} > 0, z \in \Delta\}$ by means of some results of Ruscheweyh [6]. The latter problem turns out to be equivalent to the former one with $g(z) \equiv z$.

We approach the problem of finding the radius of convexity of $K(\sigma, \lambda)$ by first determining

$$M(\beta, r) = \min_{t \in \mathbf{R}} \{M(\beta, r, t)\}, \tag{1}$$

where

$$M(\beta, r, t) = \min_{\substack{p \in P \\ |z| \leqslant r}} \left\{\operatorname{Re} \frac{zp'(z)}{p(z) + \beta + it}\right\}, \tag{2}$$

for $0 \leqslant r < 1$, $\beta > 0$ and P denoting the class of regular functions $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$ and $\operatorname{Re}\{p(z)\} > 0$ for $z \in \Delta$. This extremal problem is well known and interesting on its own. Robertson [5], by means of a variational

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method, showed that the extremal functions in (1) have the form

$$p_0(z) = \gamma \frac{1 + ze^{i\theta_1}}{1 - ze^{i\theta_1}} + (1 - \gamma)\frac{1 + ze^{i\theta_2}}{1 - ze^{i\theta_2}}$$

where $0 \leqslant \gamma \leqslant 1$, $0 \leqslant \theta_1 \leqslant 2\pi$, $0 \leqslant \theta_2 \leqslant 2\pi$ and the parameters γ, θ_1, θ_2 are to be determined. Zmorovič [8] investigated $M(\beta, r, t)$ using Robertson's result but was only able to determine $M(0, r, t)$ and $M(\beta, r, 0)$ while Robertson [5] has previously found $M(0, r)$. We obtain $M(\beta, r)$ by minimizing the functional first with respect to t and then over the class P.

2. The extremal problem. Before stating the main result of this section we need three lemmas. Using bilinear maps we may easily prove

LEMMA 1. *Let a and b be complex numbers such that* Re $b > 0$ *then*

$$\mathrm{Re}\left\{ \frac{a}{b + it} \right\} \geqslant \frac{\mathrm{Re}\{a\} - |a|}{2\mathrm{Re}\{b\}} \qquad (t \in \mathbf{R}).$$

Next, let H be the set of functions $f(z)$ regular in Δ. A real functional ϕ defined on a convex subset F of H is said to be convex on F if

$$\phi(\gamma x + (1 - \gamma)y) \leqslant \gamma\phi(x) + (1 - \gamma)\phi(y),$$

for any γ, $0 \leqslant \gamma \leqslant 1$, and x, y in F. If equality always holds then ϕ is said to be affine on F.

A function $f_0 \in F$ is said to be an extreme point of F if we cannot write $f_0 = \gamma f_1 + (1 - \gamma)f_2$, for some γ, $0 < \gamma < 1$, and distinct functions f_1, f_2 in F. Let E_F denote the set of extreme points of F. Then we have

LEMMA 2. [4]. *Let* $-\phi$ *be a convex real functional on a compact convex subset* F *of* H. *Then there always exists* $f_1 \in E_F$ *such that* $\min_{f \in F} \phi(f) = \phi(f_1)$.

The next result is a generalization of a theorem by Ruscheweyh [7].

LEMMA 3. *Let* F *be a compact convex set in* H. *Suppose that the real functionals* $-\phi$ *and* ψ *are convex and affine on* F *respectively. If* $\psi(f) > 0$ *for all* $f \in F$, *then* $\min_F\{\phi(f)/\psi(f)\} = \{\phi(f_1)/\psi(f_1)\}$, *for some* $f_1 \in E_F$.

PROOF. Define $d = \min_F\{\phi(f)/\psi(f)\} = \phi(f_0)/\psi(f_0)$, some $f_0 \in F$. Clearly $0 \leqslant \phi(f) - d\psi(f)$ and thus $0 = \min_F\{\phi(f) - d\psi(f)\}$. But $d\psi(f) - \phi(f)$ is convex and hence by Lemma 2, there exists $f_1 \in E_F$ such that $0 = \phi(f_1) - d\psi(f_1)$, and so $d = \phi(f_1)/\psi(f_1)$.

THEOREM 1. *For* $\beta > 0, p \in P, t \in \mathbf{R}, |z| \leqslant r$,

$$\text{(a)} \quad \mathrm{Re}\left\{ \frac{zp'(z)}{p(z) + \beta + it} \right\} \geqslant \frac{-2r}{(1 + r)[\beta(1 + r) + (1 - r)]}, \quad \textit{for } r \leqslant r_0,$$

$$\text{(b)} \quad \mathrm{Re}\left\{ \frac{zp'(z)}{p(z) + \beta + it} \right\} \geqslant \frac{(1 + r)^2[(1 - r)^2 - u_1]}{2u_1(1 - r^2 + \beta u_1)}, \quad \textit{for } r \geqslant r_0,$$

where $u_1 = (1-r)^2[1 + \{1 + (1+r)/\beta(1-r)\}^{1/2}]$, *and* $r_0 = (x_0 - 1)/(x_0 + 1)$, x_0 *being the positive root of the equation* $\beta x^3 - 2\beta x - 1 = 0$.

For $-1 < \beta \leqslant 0, p \in P, t \in \mathbf{R}, |z| \leqslant r < (1+\beta)/(1-\beta)$, (a) *also holds.*

PROOF. If $\operatorname{Re}\{p(z) + \beta\} > 0$ then from Lemma 1

$$\min_{t \in \mathbf{R}} \operatorname{Re}\left\{\frac{zp'(z)}{p(z) + \beta + it}\right\} = \frac{\operatorname{Re}\{zp'(z)\} - |zp'(z)|}{2\operatorname{Re}\{p(z) + \beta\}}.$$

It can be verified that $-\phi(p) \equiv |zp'(z)| - \operatorname{Re}\{zp'(z)\}$ is convex on P and that $\psi(p) \equiv 2\operatorname{Re}\{p(z) + \beta\}$ is affine on P. Also, $\psi(p) > 0$ for all $p \in P$, $|z| \leqslant r < 1$ if $\beta \geqslant 0$, and $\psi(p) > 0$ for all $p \in P$, $|z| \leqslant r < (1+\beta)/(1-\beta)$ if $-1 < \beta \leqslant 0$. Since P is compact and convex, we may apply Lemma 3 to obtain

$$\min_{p \in P} \frac{\operatorname{Re}\{zp'(z)\} - |zp'(z)|}{2\operatorname{Re}\{p(z) + \beta\}} = \frac{\operatorname{Re}\{zp_1'(z)\} - |zp_1'(z)|}{2\operatorname{Re}\{p_1(z) + \beta\}},$$

for some $p_1(z) \in E_p$. But

$$E_p = \left\{\frac{1 + ze^{i\theta}}{1 - ze^{i\theta}} : 0 \leqslant \theta \leqslant 2\pi\right\}.$$

Thus,

$$M(\beta, r) = \min_{0 \leqslant \theta \leqslant 2\pi} \frac{r(1+r)^2(\cos\theta - 1)}{[\beta(1 - 2r\cos\theta + r^2) + 1 - r^2][1 - 2r\cos\theta + r^2]}.$$

On making the substitution $u = 1 - 2r\cos\theta + r^2$, we can write

$$M(\beta, r) = \min_{m \leqslant u \leqslant n} s(u),$$

where $m = (1-r)^2$, $n = (1+r)^2$ and

$$s(u) = n(m - u)/2u(1 - r^2 + \beta u).$$

Now

$$s'(u) = \frac{n}{2} \cdot \frac{\beta u^2 - 2m\beta u - m(1 - r^2)}{u^2(1 + \beta u - r^2)^2}.$$

For $-1 < \beta \leqslant 0$ and $r < (1+\beta)/(1-\beta)$, $s'(u) < 0$, therefore in this case we find

$$M(\beta, r) = s(n) = -\frac{2r}{(1+r)[\beta(1+r) + (1-r)]}.$$

For $\beta > 0$, $s'(u)$ has zeros at

$$\begin{matrix} u_1 \\ u_2 \end{matrix} \equiv m \pm [m^2 + m(1 - r^2)/\beta]^{1/2}.$$

Since $u_2 < 0 < m$ and $u_1 > m$, $M(\beta, r) = s(u_1)$ if $u_1 \leqslant n$, otherwise $M(\beta, r)$

$= s(n)$. The condition $u_1 \geqslant n$ is equivalent to

$$(1-r)^2 + \left[(1-r)^4 + (1-r)^2(1-r^2)/\beta\right]^{1/2} \geqslant (1+r)^2$$

which, on putting $x = (1+r)/(1-r) \geqslant 1$, becomes

$$1 + (1 + x/\beta)^{1/2} \geqslant x^2.$$

Thus $(1 + x/\beta)^{1/2} \geqslant x^2 - 1 \geqslant 0$ and consequently $u_1 \geqslant n$ if $f(x) = \beta x^3 - 2\beta x - 1 \geqslant 0$. Our result follows from the fact that $f(x)$ has exactly one positive root, say x_0, in $[1, \infty)$.

3. Radius of convexity of $K(\sigma, \lambda)$.

THEOREM 2. *The radius of convexity r_c of $K(\sigma, \lambda)$ is given by*
(a) *the least positive root of*

$$0 = \sigma + (1-\sigma)\frac{1-r}{1+r} - \frac{2r}{(1+r)[\beta(1+r) + (1-r)]} \qquad (\beta = \lambda/(1-\lambda)) \quad (3)$$

when $\lambda \leqslant \lambda_0 = (x^3 - 2x + 1)^{-1}$, *where* $x = \sigma + (\sigma^2 - 2\sigma + 4)^{1/2}$,
(b) *the least positive root of*

$$0 = \sigma + (1-\sigma)\frac{1-r}{1+r} + \frac{(1+r)^2}{2}\,\frac{(1-r^2-u_1)}{u_1(1-r^2+\beta u_1)} \qquad (4)$$

when $\lambda \geqslant \lambda_0$, *where* $u_1 = (1-r)^2[1 + (1 + (1+r)/\beta(1-r))^{1/2}]$. *These results are sharp.*

PROOF. We have that $f(z) \in K(\sigma, \lambda)$ if and only if $[zf'(z)/g(z) - \lambda]/(1-\lambda)$ is subordinate to

$$\frac{1+cz}{1-z} = \frac{1}{2}(1-c) + \frac{1}{2}(1+c)\frac{1+z}{1-z},$$

where $c = e^{-2i\alpha}$. Hence, there exists $p(z) \in P$ such that

$$f'(z) = z^{-1}g(z)\left\{\lambda + (1-\lambda)\left[\tfrac{1}{2}(1-c) + \tfrac{1}{2}(1+c)p(z)\right]\right\}.$$

We thus find

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zg'(z)}{g(z)} + \frac{zp'(z)}{(1-c)/(1+c)(1-\lambda) + \lambda/(1-\lambda) + p(z)}.$$

Now as $c = e^{-2i\alpha}$

$$\frac{\lambda}{1-\lambda} + \frac{1-c}{1+c}\,\frac{1}{1-\lambda} = \beta + it,$$

where $\beta = \lambda/(1-\lambda)$ and $t = (\tan\alpha)/(1-\lambda)$. It is clear that all possible values of t are taken as α varies in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. Thus

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} = \operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} + \operatorname{Re}\left\{\frac{zp'(z)}{p(z) + \beta + it}\right\}. \qquad (5)$$

For $g(z) \in S^*(\sigma)$, it is well known that

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \geqslant \sigma + (1-\sigma)\frac{1-r}{1+r}. \tag{6}$$

From (5), (6) and Theorem 1 we obtain the required result. It remains to determine λ_0. When $\lambda = \lambda_0$, we must have that the radii of convexity in the two cases coincide, say $r_c = R$. Then, according to Theorem 1 and part (a) of this theorem, the values of R and $\beta_0 = \lambda_0/(1-\lambda_0)$ are determined by

$$\beta_0 x^3 - 2\beta_0 x - 1 = 0, \qquad x = (1+R)/(1-R) > 1,$$

and

$$0 = \sigma + (1-\sigma)\frac{1-R}{1+R} - \frac{2R}{(1+R)[\beta_0(1+R) + (1-R)]}.$$

Elimination of β_0 and R from these equations gives

$$(x^2 - 2\sigma x + 2\sigma - 4) = 0.$$

It follows that $x = \sigma + (\sigma^2 - 2\sigma + 4)^{1/2}$ and hence λ_0 is (uniquely) determined and the proof of Theorem 2 is completed.

REMARK 1. For $\lambda \leqslant \lambda_0$, the extremal function is

$$f_0(z) = \int_0^z \left\{\lambda + (1-\lambda)\frac{1+\xi}{1-\xi}\right\} \frac{d\xi}{(1-\xi)^{2(1-\sigma)}}.$$

For $\lambda \geqslant \lambda_0$, the extremal functions are

$$f_0(z) = \int_0^z \left\{\lambda + \frac{1-\lambda}{2}\left[(1-c_0) + (1+c_0)\frac{1+e^{i\theta_0}\xi}{1-e^{i\theta_0}\xi}\right]\right\}\frac{d\xi}{(1-\xi)^{2(1-\sigma)}}$$

where $\cos\theta_0 = (1 + r^2 - u_1)/2r$ and $c_0 = [(1-\lambda)it_0 - 1]/[(1-\lambda)it_0 + 1]$ with u_1 as defined in Theorem 1 and

$$t_0 = \frac{2r\sin\theta_0}{u_1}\left\{\frac{(1 - r^2 + \beta u_1)(1-r)}{(1+r)^2[(1-r)^2 - u_1]} - 1\right\}.$$

Putting $g(z) \equiv z$, we recover Jankovics' result [2]. For $\sigma = 0$, our result differs from that of Jablonski and Wesolowski [1] as expected.

REMARK 2. Libera [3] considered the class $C(\sigma, \lambda)$ of functions $f(z)$ such that $\operatorname{Re}\{e^{i\alpha}zf'(z)/g(z)\} > \lambda$ for some $g(z) \in S^*(\sigma)$. Libera was only able to place a lower bound on the radius of convexity of $C(\sigma, \lambda)$. Now, since $C(\sigma, \lambda) \subseteq K(\sigma, \lambda)$, r_c for $K(\sigma, \lambda)$ is another lower bound for the radius of convexity of $C(\sigma, \lambda)$. However, for the case $\lambda \leqslant \lambda_0$, as the extremal function $f_0(z)$ belongs to $C(\sigma, \lambda)$, we conclude that r_c is also the radius of convexity for $C(\sigma, \lambda)$.

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