PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 72, NUmber 2, November 1978

BOOLEAN ALGEBRAS WITHOUT NONTRIVIAL ONTO ENDOMORPHISMS EXIST IN EVERY UNCOUNTABLE CARDINALITY

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ABSTRACT. We prove, assuming ZFC, that for every uncountable cardinal λ, there is a Boolean algebra of cardinality λ, without onto endomorphisms other than the identity.

1. Introduction. In this note we construct Boolean algebras (hereafter denoted by BA's), in order to prove the following theorem.

THEOREM 1.1. *For every uncountable cardinal* λ, *there is a BA* B *of power* λ, *such that* B *does not have onto endomorphisms except the identity.*

Of the numerous results about rigid BA's let us mention four. Shelah **[S1]** proved that for every uncountable cardinal λ, there is a rigid (that is, without automorphisms except the identity) BA of cardinality λ. Bonnet **[B1]**, assuming CH, constructed a BA of power continuum, without onto or 1-1 endomorphisms except the identity. Loats [L] and independently Bonnet **[B2]** generalized Bonnet's construction to κ^+, assuming of course $\kappa^+ = 2^\kappa$.

Every BA has some trivial endomorphisms. Let us describe them. Let B be a BA, $a_1, \ldots, a_n \in B$ and for every $1 \leqslant i < j \leqslant n$, $a_i \cap a_j = 0 \neq a_i$, and $\bigcup_{i=1}^n a_i = 1$. Let $B \restriction a_i$ be the BA that B induces on $\{x | x \subseteq a_i\}$, and for every $1 \leqslant i \leqslant n$ let F_i be an ultrafilter on $B \restriction a_i$. Let B' be the power set of $\{1, \ldots, n\}$, so B' is a BA. Let $\sigma \in B'$, and let π be an endomorphism of B' such that for every $\sigma_1 \subseteq \sigma$, $\pi(\sigma_1) \supseteq \sigma_1$. Let f: $B \to B$ the unique endomorphism for which: (1) for every $i \notin \sigma$ and for every $x \subseteq a_i$: if $x \in F_i$, then $f(x) = \bigcup \{a_j | j \in \pi(i)\}$, and if $x \notin F_i$, then $f(x) = 0$; (2) for every $i \in \sigma$ and for every $x \subseteq a_i$: if $x \in F_i$, then $f(x) = x \cup \bigcup \{a_j | j \in \pi(i) - \{i\}\}$, and if $x \notin F_i$, then $f(x) = x$.

Let us call such endomorphisms inevitable.

Shelah **[S2]** has found this full set of inevitable endomorphisms, and proved that if $\diamondsuit_{\aleph_1}$ holds, then there is a BA of power $\aleph_1$ with only inevitable endomorphisms.

So at present the following is known.

Presented to the Society, August 17, 1977 under the title *On 'strongly' rigid Boolean algebras and cofinalities of* BA*'s*; received by the editors November 7, 1977.

AMS (*MOS*) *subject classifications* (1970). Primary 02J10; Secondary 02H15.

[1]This constitutes a chapter of the first author's Ph. D. Thesis, prepared at the University of Colorado under the supervision of Professor J. D. Monk.

(1) ZFC $\Rightarrow$ "For every $\lambda > \aleph_0$, there is a BA of power λ, without onto endomorphisms except the identity."

(2) $\lambda^+ = 2^\lambda \Rightarrow$ "There is a BA of power λ^+ without onto or 1-1 endomorphisms except the identity."

(3) $\diamondsuit_{\aleph_1} \Rightarrow$ "There is a BA without noninevitable endomorphisms."

Whether ZFC is sufficient for (2), and whether, say ZFC + CH is sufficient for (3) is open.[2]

For BA's with few order preserving functions, see Rubin **[Ru]** and Shelah **[S2]**.

Finally let us remark about countable BA's. Every countable BA has $2^{\aleph_0}$ automorphisms. See e.g. Monk **[M]**. Loats in **[L]** showed also that every countable BA has $2^{\aleph_0}$ onto endomorphisms which are not automorphisms.

The proof of Theorem 1.1 combines methods of Shelah **[S3]** and Reiger **[R]**.

Shelah (unpublished) proved in ZFC that for every uncountable cardinal λ, there is a rigid dense linear ordering of power λ. This was done by tagging every element of the linear ordering by a stationary set, more precisely, by an element of the BA $D(\lambda) = P(\lambda)/I(\lambda)$, where $I(\lambda)$ is the ideal of all subsets of λ, that are disjoint from some closed and unbounded subset of λ. Reiger **[R]** used the fact that if $f\colon X \to Y$ is a continuous 1-1 mapping, $x \in X$ and there is a 1-1 sequence of order type λ converging to x, then there is such a sequence converging to $f(x)$. He thus constructed a Stone space of a BA in which the elements are tagged by their cofinalities. In order that each element would have been tagged by a different cofinality he had to assume that the Stone space has cardinality $\aleph_\alpha = \alpha$. We noticed that much of Shelah's tagging is preserved under 1-1 continuous functions of the order topology, so we had more tags, and could construct Stone spaces in more cardinalities than Rieger.

2. The construction.

DEFINITION. Let X be a topological space and $x \in X$. Then $\mathrm{Cf}(x, X) = \{\mu \mid \mu$ is a regular infinite cardinal and there is a sequence $\{x_i \mid i < \mu\}$ in X such that $x = \lim_{i<\mu} x_i$ and for every $\alpha < \mu$, $\lim_{i<\alpha} x_i$ exists and is different from $x\}$. When no confusion might be caused we omit X and write $\mathrm{Cf}(x)$.

DEFINITION. Let X be a topological space, $x \in X$ and μ be an uncountable regular cardinal; we say that x is *μ-special* in X, if $\mu \in \mathrm{Cf}(x)$, and for every $\{x_i \mid i < \mu\}$, $\{y_i \mid i < \mu\}$ as in the definition of $\mathrm{Cf}(x, X)$, $\{\alpha \mid \lim_{i<\alpha} x_i = \lim_{i<\alpha} y_i\}$ is closed and unbounded in μ. From now on let λ be a fixed regular uncountable cardinal. If X is a topological space and $x \in X$ is λ-special, we define S_x^X to be the element of $D(\lambda)$ gotten in the following way. Let $\{x_i \mid i < \lambda\}$ be as in the definition of $\mathrm{Cf}(x, X)$ and let $S' = \{\alpha \mid \lambda \in \mathrm{Cf}(\lim_{i<\alpha} x_i)\}$. Let $S_x^X = S'/I(\lambda)$. Since x is λ-special, S_x^X is independent of the choice of $\{x_i \mid i < \lambda\}$. When no confusion may arise, we write S_x instead of

[2] ADDED IN PROOF. New results by Monk and Shelah answer the latter question positively.

S_x^X. In the sequel we will identify elements of $D(\lambda)$ and their representatives in $P(\lambda)$.

Let $S(B)$ denote the Stone space of the BA B.

LEMMA 2.1. *A BA B does not have onto endomorphisms except the identity iff $S(B)$ does not have* 1-1 *continuous functions except the identity.*

PROOF. Trivial.

LEMMA 2.2. *If X is a topological space, f: $X \to X$ is* 1-1 *and continuous and $x \in X$, then*: (a) $\mathrm{Cf}(x) \subseteq \mathrm{Cf}(f(x))$; (b) *if $x, f(x)$ are λ-special, then $S_x \subseteq S_{f(x)}$.*

PROOF. Trivial.

When we refer to a linear ordering as a topological space, we always mean the order topology.

If X is Hausdorff compact totally disconnected space, then $B(X)$ will denote the BA of clopen subsets of X; so $X \cong S(B(X))$. If I is a complete linear ordering then I is compact.

If I is a linear ordering, $x \in I$, and x is a left limit (that is, x is not a successor), let $\mathrm{cf}^-(x, I)$ be the unique regular cardinal μ such that there is a strictly increasing sequence of order type μ converging to x; if x is not a left limit, then $\mathrm{cf}^-(x)$ is undefined. $\mathrm{cf}^+(x, I)$ is defined similarly. It is clear that if I is a complete linear ordering, $x \in I$, then $\mathrm{cf}(x, I) = \{\mathrm{cf}^-(x, I), \mathrm{cf}^+(x, I)\}$, where undefined objects are omitted; and x is λ-special iff either $\mathrm{cf}^-(x, I) = \lambda$ and $\mathrm{cf}^+(x, I) \neq \lambda$ or is undefined, or $\mathrm{cf}^+(x, I) = \lambda$ and $\mathrm{cf}^-(x, I) \neq \lambda$ or is undefined.

Let us describe now the aim of our construction for regular cardinals.

LEMMA 2.3. *Let I be a linear ordering with the following properties*: (1) *I is complete*; (2) $|\{x | x \in I$ *and x has a successor in* $I\}| = \lambda$; (3) *the set of points of I that have a successor in I is dense in I, that is*: *if the open interval (y, z) is nonempty, then there is $x_1 \in (y, z)$ such that x_1 has a successor*; (4) *for every $x \in I$ either $\lambda \notin \mathrm{Cf}(x)$, or x is λ-special*; (5) *there is a dense subset $P \subseteq I$, such that*: *for every $x \in P$, x is λ-special and $S_x \neq 0$*; *for every $x, y \in P$, if $x \neq y$, then $S_x \cap S_y = 0$*; *and if $x \in I - P$, then either $\lambda \notin \mathrm{Cf}(x)$ or else $S_x = 0$.*

Then: *I is Hausdorff compact and totally disconnected, $|B(I)| = \lambda$, and $B(I)$ does not have onto endomorphisms except the identity.*

PROOF. Let I satisfy conditions (1)–(5). Since I is complete, it is compact. By (3) it is clear that every two distinct elements of I can be separated by a set of the form $V_x = \{y | y > x\}$ where x has a successor in I, and clearly V_x is clopen, so I is totally disconnected. It is easy to see that $B(I)$ is the BA generated by $\{V_x | x$ has a successor in $I\}$, so $|B(I)| = \lambda$. Suppose now by contradiction, that there is a 1-1 continuous function h from I to I, different from the identity. Since I is Hausdorff and P is dense in I there is $x \in P$ such that $h(x) \neq x$. Since by (5) x is λ-special, $\lambda \in \mathrm{Cf}(x)$; so by Lemma 2.2 (a)

$\lambda \in \mathrm{Cf}(h(x))$. By (4) $h(x)$ is λ-special, so by 2.2 (b) $S_x \subseteq S_{h(x)}$. However, again by (5), $S_x \neq 0$ and $S_{h(x)}$ is either 0 or it is disjoint from S_x, contradicting the fact that $S_x \subseteq S_{h(x)}$. So the identity is the only 1-1 continuous mapping from I to I, and by Lemma 2.1, the identity is the only onto endomorphism of $B(I)$. Q.E.D.

Now for every regular cardinal λ we are going to construct a linear ordering I as in Lemma 2.3.

Let $\mathbf{Z}$ denote the linear ordering of the integers. If I and J are linear orderings, let $I + J$ denote their sum. If I is a linear ordering and for every $i \in I$, K_i is a linear ordering, let $\Sigma_{i \in I} K_i$ denote the sum of the K_i's over I. If I is a linear ordering let I^* be the reversed linear ordering. λ is considered as a linear ordering, where the ordering relation is $\in$.

We will first describe the construction of I, and then list without a proof a series of easy observations, that will lead to the conclusion that I has properties (1)–(5) of Lemma 2.3.

LEMMA 2.4. *The construction of I.*

Suppose $S \subseteq \lambda$ is a set of limit ordinals. For every $i \in Z + \lambda$ let us define the linear ordering J_i as follows. If $i \in S$ let $J_i = 1 + \lambda^*$, where 1 is the linear ordering with exactly one element; if $i \notin S$ let $J_i = 1$. Let $I_S = \Sigma_{i \in Z+\lambda} J_i$.

If I is a linear ordering let E_I be the equivalence relation on I defined as follows: $x\, E_I\, y$ iff there are just finitely many elements between x and y. Every equivalence class of E_I is convex.

If $x \in I$, let x/E_I denote the equivalence class of x, and let $I/E_I = \{x/E_I | x \in I\}$.

OBSERVATION 1. If $A \in I_S/E_{I_S}$ then A is either of order type ω, or ω^*, or Z, or $|A| = 1$.

We now define $P_S \subseteq I_S$. It suffices to define $P_S \cap A$ for every $A \in I_S/E_{I_S}$. So let $A \in I_S/E_{I_S}$: if $|A| = 1$, then $P_S \cap A = \varnothing$; otherwise let B be either ω or Z and f: $B \to A$ be an onto order preserving or order reversing function; define $P_S \cap A = \{f(2i + 1) | i \in B\}$. This defines P_S up to isomorphism of (I_S, P_S).

We now construct linear orderings I_n for every $n < \omega$. At the same time we define $P_n \subseteq I_n$.

Let $\{S_{in} | i < \lambda, n < \omega\}$ be a family of pairwise disjoint stationary subsets of λ.

Let $I_0 = 1 + I_\varnothing + 1$, $P_0 = P_\varnothing$ and let us denote $I_{-1} = \{\min(I_0), \max(I_0)\}$.

Suppose I_n and P_n have been defined. Let f_n: $P_n \to \{S_{in} | i < \lambda\}$ be a 1-1 function. For every $x \in P_n$, let $K_x = I_{f_n(x)} + 1$, and for every $x \in I_n - P_n$, let $K_x = 1$. Let $I_{n+1} = \Sigma_{x \in I_n} K_x$. W.l.o.g. we identify I_n with a subset of I_{n+1} according to the following embedding g: $g(x) = \max(K_x)$, $x \in I_n$. Let P_{n+1} be the subset of I_{n+1} which satisfies: $P_{n+1} \cap I_n = \varnothing$ and for every $x \in P_n$, $P_{n+1} \cap K_x = P_{f_n(x)}$.

Let $I_\omega = \bigcup_{n<\omega} I_n$, and I be the completion of I_ω (under Dedekind cuts). Let $P = \bigcup_{n<\omega} P_n$. This concludes the definition of I and P.

OBSERVATION 2. (a) P_n does not contain limit points. (b) If $x, y \in I_n - I_{n-1}$, and y is the successor of x, then $x \in P_n$ iff $y \notin P_n$.

OBSERVATION 3. Let $x \in I_n$ and $J \in \{I_k | n < k \leqslant \omega\} \cup \{I\}$. Then: (a) If $x \in P_n$ and y is the successor of x in I_n then y is the successor of x in J. (b) If $\lim_{i<\alpha} x_i = x$ in I_n, then $\lim_{i<\alpha} x_i = x$ in J.

OBSERVATION 4. If x is a limit point I_n, then x is a limit of a sequence of elements which belong to P_n.

PROOF. By induction on n, then distinguish between the following cases: $x \in I_n - I_{n-1}$; $x \in P_{n-1}$; $x \in I_{n-1} - P_{n-1}$.

OBSERVATION 5. Let $x \in I_n$ and $J \in \{I_k | n < k \leqslant \omega\} \cup I$. Then: (a) x is a limit point in J; (b) if x is a limit point of I_n then $\lambda \notin \mathrm{Cf}(x, J) - \mathrm{Cf}(x, I_n)$.

OBSERVATION 6. (a) I_n is a complete linear ordering. (b) If $x \in I - I_\omega$, then $\mathrm{cf}^-(x, I) = \mathrm{cf}^+(x, I) = \aleph_0$.

CONCLUSION 7. I satisfies (1), (2), (3) of Lemma 2.3.

PROOF. (1) holds by the definition of I. (2) holds by Observations 3(a) and 6. (3) holds by Observations 3(a) and 4.

OBSERVATION 8. (a) If $x \in P_n$, then x is λ-special in I and $S_x^I = f_n(x)/I(\lambda)$. (b) If $x \in I_n - I_{n-1} - P_n$, then either $\lambda \notin \mathrm{Cf}(x, I)$, or $\mathrm{cf}^+(x, I) = \lambda$, $\mathrm{cf}^-(x, I) < \lambda$ and $S_x^I = 0$.

CONCLUSION 9. I satisfies (4) and (5) from Lemma 2.3.

For λ regular, Theorem 1.1 now follows from Lemma 2.3 and Conclusions 7 and 9.

Let us denote by I_λ the linear ordering that we have constructed in 2.4 for λ. If μ is a limit cardinal, let $\mu = \sum_{i<\kappa} \mu_i$ where $\{\mu_i | i < \kappa\}$ is a strictly increasing sequence of regular cardinals. Let $I_\mu = (\sum_{i<\kappa} I_{\mu_i}) + 1$. It is easy to see that I_μ is complete and totally disconnected, $|B(I_\mu)| = \mu$, and the identity is the only 1-1 continuous mapping from I_μ to I_μ.

So Theorem 1.1 is proved.

REMARK. Of course for every $\kappa > \aleph_0$, we can construct in the above method a family of 2^κ BA's as in Theorem 1.1, such that there is no homomorphism from one BA in the family onto another.

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