AN ELEMENTARY SOLUTION OF THE MONOTONE MAPPING PROBLEM

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ABSTRACT. A simple example of a monotone, noncompact mapping from $\mathbf{R}^3$ to $\mathbf{R}^3$ is constructed.

We let $\mathbf{R}^n$ denote Euclidean n -dimensional space with the usual topology, and we use the word *map* as an abbreviation for continuous function.

A map is *monotone* if and only if point inverses are compact and connected, and is *compact* if and only if inverse images of compact sets are compact.

In 1959, Whyburn [5] proved that any monotone map from $\mathbf{R}^2$ onto itself is necessarily compact, and asked if the same result held for $\mathbf{R}^n$. In 1969, a negative answer for $n \geq 4$ was given by Glaser [4], and later that year, R. H. Bing [1] produced a monotone, noncompact map from $\mathbf{R}^3$ onto itself, completely settling the question. Bing's argument is ingenious and difficult. The idea is roughly as follows: One-to-one maps from $\mathbf{R}_+^3$ (the closed upper half of $\mathbf{R}^3$) onto $\mathbf{R}^3$ were known ([2], [3]). Such maps are necessarily noncompact. Then if one could find a monotone map from $\mathbf{R}^3$ onto $\mathbf{R}_+^3$, its composition with a one-to-one map from $\mathbf{R}_+^3$ onto $\mathbf{R}^3$ would produce a monotone, noncompact map from $\mathbf{R}^3$ onto itself. In order to construct a monotone map from $\mathbf{R}^3$ onto $\mathbf{R}_+^3$, Bing proved the following lemma, which is the crux of the matter.

LEMMA. *There is a map f from the 3-cell D , onto itself, with the following properties:*

- (1) *f is monotone,*
- (2) *if $y \in D$, then $f^{-1}(y)$ intersects the boundary of D , and*
- (3) *there is a point q on the boundary of D whose inverse image is a single point.*

On the basis of the above lemma it is a relatively easy matter to produce the desired map. Consider the 3-sphere S^3 as the union of two copies of I^3 attached along their boundaries by the identity map. Apply the map f of the lemma to each copy of I^3 . This yields a monotone map from S^3 onto I^3 which induces a monotone map from $S^3 - \{f^{-1}(q)\} (\cong \mathbf{R}^3)$ onto $I^3 - \{q\} (\cong \mathbf{R}_+^3)$.

The purpose of this paper is to provide an elementary proof of the

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preceding lemma. We would like to thank Professors R. H. Bing and C. E. Burgess for a careful reading of the first draft of this paper. They pointed out an error and suggested several improvements which we have incorporated into this final manuscript.

PROOF OF LEMMA. *Step 0.* We begin with a trivial modification of the usual Cantor function. Let C denote the usual middle thirds Cantor set in $[0, 1]$. The function $g: C \rightarrow [0, 1]$ defined by

$$g\left(\sum_{n=1}^{\infty} \frac{a_n}{3^n}\right) = \sum_{n=1}^{\infty} \frac{a_n}{2^{n+1}}$$

has a unique continuous extension to a map $g_0: [-1, 2] \rightarrow [0, 1]$ which is constant on each subinterval of $[-1, 2] - C$. g_0 has the following properties:

- (a) g_0 is monotone,
- (b) if $y \in [0, 1]$, then $g_0^{-1}(y)$ intersects C .

Step 1. Let $g_1: [-1, 2]^3 \rightarrow [0, 1]^3$ be defined by $g_1(x, y, z) = (g_0(x), g_0(y), g_0(z))$. g_1 has the following properties:

- (a) g_1 is monotone,
- (b) if $y \in [0, 1]^3$, then $g_1^{-1}(y)$ intersects $C \times C \times C$.

Step 2. Let K denote the Cantor set $C \times C \times C \subset \text{Int}([-1, 2]^3)$. K is a tame Cantor set. Let g_2 be a homeomorphism from $[-1, 2]^3$ onto itself such that $g_2^{-1}(K) = \{0\} \times C \times \{0\}$, and which is the identity on the boundary.

Step 3. In this step we construct a map g_3 from $[0, 1]^3$ onto $[-1, 2]^3$ such that

- (a) g_3 is monotone,
- (b) $g_3^{-1}(\{0\} \times C \times \{0\})$ is a subset of the boundary of $[0, 1]^3$, and
- (c) g_3 takes the left and right faces of $[0, 1]^3$ homeomorphically onto the left and right faces respectively of $[-1, 2]^3$ so that $g_3(0, y, z)$ and $g_3(1, y, z)$ have the same last two coordinates.

Consider the following diagram.

Let B denote the 3-cell which is the complement of the cone shaped tunnel T in $[-1, 2]^3$. Let h be a homeomorphism from $[0, 1]^3$ onto B which satisfies (c). Notice that horizontal planes intersect the boundary of T in simple closed

curves. Let j be the map from B onto $[-1, 2]^3$ which shrinks each of these simple closed curves to points. Then the map g_3 is $(j \circ h)$.

Step 4. Let g be the map from $[0, 1]^3$ onto $[0, 1]^3$ defined by $g = (g_1 \circ g_2 \circ g_3)$. Then

(a) g is monotone and

(b) if $y \in [0, 1]^3$ then $g^{-1}(y)$ intersects the boundary of $[0, 1]^3$.

At this point we have a map satisfying conditions (1) and (2) of the lemma. To satisfy (3) we glue countably many copies of g together as in the following diagram.

This completes the proof.

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