DIFFERENTIABLE PSEUDO-FREE CIRCLE ACTIONS ON HOMOTOPY SPHERES¹

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ABSTRACT. Let G denote the circle group, φ a differentiable pseudo-free G -action of type $(p_1, \dots, p_k)$ on a homotopy sphere Σ^{2n+1} , and X the vector field induced by φ . If w is a G -invariant 1-form satisfying $w(X) = 1$, then we will prove that $\int_{\Sigma} w \wedge (dw)^n = \pm (p_1 p_2 \cdots p_k)^{-1}$.

Let R denote the set of real numbers, Z the set of integers, and $G = R/Z$ —the circle group.

Suppose we have an effective G -action without fixed point on a smooth compact manifold M^{2n+1} . This action induces a vector field X of period 1 on M . Let p denote the least common multiple of the orders of the isotropy groups of the elements of M . If w is a G -invariant 1-form such that $w(X) = 1$, then Weinstein [9] showed that $p^n \int_M w \wedge (dw)^n$ is an integer. The aim of this paper is to compute Weinstein's integers in certain cases.

An effective G -action φ on a homotopy sphere Σ^{2n+1} is called pseudo-free of type $(p_1, p_2, \dots, p_k)$, if there exist k isolated exceptional orbits such that $Z/p_i Z$ (the cyclic group of order p_i) is the isotropy group for the points on the i th orbit, and the integers $p_1, \dots, p_k$ are relatively prime to one another, [5], [6]. In the following, we let $p = p_1 p_2 \cdots p_k$.

Let S^1 denote the group of complex numbers (under multiplication) of absolute value 1. Let S^{2n+1} be the unit sphere in the complex $(n+1)$ -space $\mathbb{C}^{n+1}$. Then there exists a linear pseudo-free S^1 -action ψ_p on S^{2n+1} , given by the equation

$$g(z_0, z_1, \dots, z_n) = (g^p z_0, g z_1, \dots, g z_n).$$

In order to use the formula in [9], we construct another G -action φ_p , which is induced from ψ_p by the map $G \rightarrow S^1$ sending g to $e^{2\pi i g}$.

PROPOSITION. *Let X denote the vector field on $S = S^{2n+1}$ induced by φ_p , and let w be a G -invariant 1-form satisfying $w(X) = 1$. Then we have $\int_S w \wedge (dw)^n = \pm p^{-1}$.*

PROOF. The orbit space S/G is a V -manifold, [9]. For V -manifolds, we have a theory of differential forms, fibre bundles etc., and the deRham theorem holds, [1], [7], [8]. Furthermore, there is a Chern-Weil theory of characteristic classes, [4], [9]. In particular, if E is a principal circle bundle

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over a V -manifold, then one can compute a form representing the first Chern class $c_1(E)$ in the usual way from a connection on E .

Let $K = Z/pZ \subseteq G$, then $S/K \rightarrow S/G$ is a principal G/K bundle, and S/K is a V -manifold. The G -invariant 1-form w induces a G/K -invariant 1-form $\bar{w}$ on S/K . Also, X induces a vector field $\bar{X}$ on S/K . The vector field $p^{-1}\bar{X}$ generates the G/K -action on S/K (its flow has period 1), so $p\bar{w}$ is a connection form for the bundle $S/K \rightarrow S/G$. By the Chern-Weil theory, $c_1(S/K)$ is represented by $p\Omega_G$, where Ω_G on S/G pulls back to $d\bar{w}$ (see [9, pp. 267–268], also cf. [2]).

It was shown in [3] that S/G has the same integral cohomology as CP^n , the complex projective space. Let $\beta_1 \in H^2(S/G; Z) = Z$ be a generator. It is easy to see that the Chern class of the principal bundle $S/K \rightarrow S/G$ is $\pm \beta_1$ (constructing a map from $S/K \rightarrow S/G$ to $S^{2N+1} \rightarrow CP^N$ for some large N by considering $S^{2n+1}/G = (S^1 \times D^{2n})/G \cup (D^2 \times S^{2n-1})/G$). Thus the cohomology class $[\Omega_G]$ determined by Ω_G is equal to $\pm p^{-1}\beta_1$.

We have the following formula from [9, p. 268]:

$$\int_S w \wedge (dw)^n = \int_{S/G} \Omega_G^n.$$

Let $\beta_j \in H^{2j}(S/G; Z)$ be the generator for $1 \leq j \leq n$. It is known that we may choose β_j 's in such a way that $\beta_1^2 = p\beta_2$, $\beta_1^3 = p^2\beta_3$, $\dots$, $\beta_1^n = p^{n-1}\beta_n$ [6, p. 95]. We also use β_k to denote a $2k$ -form on S/K representing β_k . Then we have $[\Omega_G]^n = \pm p^{-n}\beta_1^n = \pm p^{-1}\beta_n$, and $\int \Omega_G^n = \pm p^{-1} \int \beta_n = \pm p^{-1}$. Q.E.D.

Suppose that φ is a differentiable pseudo-free G -action of type $(p_1, p_2, \dots, p_k)$ on a homotopy sphere Σ^{2n+1} . Then the main theorem of [5] asserts the existence of a G -equivariant map F of Σ^{2n+1} into S^{2n+1} (with the action φ_p) of degree ± 1 , that is, we have the following diagram:

$$\begin{array}{ccc} \Sigma^{2n+1} & \xrightarrow{F} & S^{2n+1} \\ \downarrow & & \downarrow \\ \Sigma/G & \xrightarrow{f} & S/G \end{array}$$

and the principal G/K -bundle $\Sigma/K \rightarrow \Sigma/G$ is induced by the homotopy equivalence $f: \Sigma/G \rightarrow S/G$. Therefore, $f^*: H^*(S/G; Z) \rightarrow H^*(\Sigma/G; Z)$ is a ring isomorphism, and we have $c_1(\Sigma/K) = \pm f^*(\beta_1)$. The proof of the proposition then yields:

THEOREM. *Let φ be a differentiable pseudo-free circle action of type $(p_1, \dots, p_k)$ on a homotopy sphere Σ^{2n+1} , and X the vector field induced by φ . If w is a G -invariant 1-form satisfying $w(X) = 1$, then we have*

$$\int_{\Sigma} w \wedge (dw)^n = \pm p^{-1}, \quad \text{where } p = p_1 p_2 \cdots p_k.$$

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