RIGHT ALTERNATIVE ALGEBRAS AND WEDDERBURN PRINCIPAL THEOREM

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ABSTRACT. Examples are given of right alternative algebras and of quadratic Jordan algebras (the latter only over fields of characteristic two) without a Wedderburn splitting but such that at least one u -isotope admits a Wedderburn splitting. An example of a noncommutative Jordan algebra over a field of characteristic $\neq 2$ is given such that no u -isotope admits a Wedderburn splitting.

Everyone who has worked with right alternative algebras knows that the defining relations are rather weak. Hence there should exist a lot of right alternative algebras. But only a few examples may be found scattered in the literature. In this note we study a family of unital right alternative k -algebras of rank 8 by giving the multiplication table of a basis that depends on 5 independent parameters. Each of our right alternative k -algebras R has a radical Z of rank 4 such that $ZZ = 0$ and R/Z is isomorphic to the full 2-rowed matrix ring over k . We determine for which values of the parameters our algebras admit a direct splitting $R = S \oplus Z$, where S is a subalgebra of R isomorphic to R/Z . As a consequence one sees that the Wedderburn Principal Theorem does not hold in the class of all right alternative algebras. But R has at least one u -isotope admitting a Wedderburn splitting. Moreover we get an example of a quadratic Jordan algebra R^+ of characteristic two that has no Wedderburn splitting. This shows that the Wedderburn Principal = Albert-Penico-Taft Theorem for linear Jordan algebras cannot be extended to quadratic Jordan algebras of characteristic two. But there is a u -isotope admitting a Wedderburn splitting. In a final section we present a noncommutative Jordan algebra with the property that no u -isotope admits a Wedderburn splitting.

Right alternative algebras. Let k be a commutative associative ring with unit element. Choose as parameters elements $\alpha, \varepsilon, \tau, \xi, \kappa$ of k with the only restriction that κ is invertible. Let R be a free k -module of rank 8 with basis denoted by $c_1, c_2, c_{12}, c_{21}, z_1, z_2, z_{12}, z_{21}$. We give R the structure of a k -algebra $R = R(\alpha, \varepsilon, \tau, \xi, \kappa)$ by requiring that these elements multiply according to the following multiplication table.

Received by the editors November 14, 1977.

AMS (MOS) subject classifications (1970). Primary 17A30, 17C50, 17A15, 17C10.

Key words and phrases. Wedderburn Principal Theorem, Albert-Penico-Taft Theorem, right alternative algebras, quadratic Jordan algebras, noncommutative Jordan algebras, u -isotope.

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	c_1	c_2	c_{12}	c_{21}
c_1	c_1	0	c_{12}	z_2
c_2	0	c_2	0	$c_{21} - z_2$
c_{12}	0	c_{12}	ϵz_1	$c_1 + \alpha z_1 - z_{21}$
c_{21}	$c_{21} - z_2$	z_2	$c_2 + \epsilon \kappa z_2 + z_{21} + \tau z_1$	$\kappa(\alpha + \tau)z_2$
z_1	z_1	0	0	$z_{12} - \kappa \tau z_1$
z_2	0	z_2	$z_{21} + \tau z_1$	0
z_{12}	$\kappa \tau z_1$	$z_{12} - \kappa \tau z_1$	z_1	$\kappa \tau (z_{12} - \kappa \tau z_1)$
z_{21}	z_{21}	0	0	$z_2 - \tau(z_{12} - \kappa \tau z_1)$

	z_1	z_2	z_{12}	z_{21}
c_1	z_1	0	$z_{12} - \kappa \tau z_1$	0
c_2	0	z_2	$\kappa \tau z_1$	z_{21}
c_{12}	0	$\kappa^{-1} z_{12} - \tau z_1$	$-\xi z_1$	$\kappa^{-1}(1 + \xi)z_1$
c_{21}	$\kappa(\tau z_1 + z_{21})$	0	$-\kappa \tau (z_{12} - \kappa \tau z_1) + \kappa(1 - \xi)z_2$	$\tau(z_{12} - \kappa \tau z_1) + \xi z_2$
z_1	0	0	0	0
z_2	0	0	0	0
z_{12}	0	0	0	0
z_{21}	0	0	0	0

MULTIPLICATION TABLE

Since k is arbitrary the algebra R is right alternative in case $(xy)y = xy^2$ and $(xy)y^2 = xy^3$ holds for all x, y in R [12]. Since $c_1 + c_2$ is a unit element of R it suffices to prove these equations for $x \in \{c_1, c_{12}, c_{21}, z_1, z_2, z_{12}, z_{21}\}$ and all $y = \eta_1 c_1 + \eta_2 c_2 + \eta_{12} c_{12} + \eta_{21} c_{21} + \xi_1 z_1 + \xi_2 z_2 + \xi_{12} z_{12} + \xi_{21} z_{21} \in R$, $\eta_2 = 0$. One has

$$y^2 = (\eta_1 + \eta_2)y + (\eta_{12}\eta_{21} - \eta_1\eta_2)(c_1 + c_2) \\ + \beta_1 z_1 + \beta_2 z_2 + \beta_{12}(z_{12} + \kappa z_{21})$$

with

$$\beta_1 := \eta_{12}^2 \epsilon + (\eta_1 - \eta_2)\xi_1 + \eta_{12}\eta_{21}(\alpha + \tau) \\ + \eta_{12}\xi_{12}(1 - \xi) + \eta_{12}\xi_{21}(1 + \xi)\kappa^{-1}, \\ \beta_2 := \eta_{12}\eta_{21}\epsilon\kappa - (\eta_1 - \eta_2)\xi_2 + \eta_{21}^2(\alpha + \tau)\kappa \\ + \eta_{21}\xi_{12}(1 - \xi)\kappa + \eta_{21}\xi_{21}(1 + \xi), \\ \beta_{12} := \eta_{12}\xi_2\kappa^{-1} + \eta_{21}\xi_1$$

satisfying the equation $\eta_{12}\beta_2 - \eta_{21}\beta_1\kappa + (\eta_1 - \eta_2)\beta_{12}\kappa = 0$. With these

remarks the reader will easily verify $(xy)y = xy^2$ and $(xy)y^2 = xy^3$. Hence for all choices of $\alpha, \varepsilon, \tau, \xi, \kappa$ R is a right alternative algebra.

Trivially $Z := kz_1 + kz_2 + kz_{12} + kz_{21}$ is an ideal of R such that $ZZ = 0$ and R/Z is the 2-rowed matrix ring over k . Each element of Z is an absolute zero divisor of R and R/Z has no absolute zero divisor besides zero. Hence Z is the (lower) radical of R [2], [13], [14], [15].

The Wedderburn Principal Theorem holds for R if and only if R contains a subalgebra S such that $R = S \oplus Z$. The canonical epimorphism of R onto R/Z restricts to an isomorphism φ of S onto R/Z . Hence the Wedderburn Principal Theorem holds for R if and only if R contains elements d_1, d_2, d_{12}, d_{21} such that $d_1 + d_2 = c_1 + c_2$, $\varphi(d_1) = c_1 + Z$, $\varphi(d_2) = c_2 + Z$, $\varphi(d_{12}) = c_{12} + Z$, $\varphi(d_{21}) = c_{21} + Z$ and $d_1^2 = d_1, d_2^2 = d_2, d_1d_{12} = d_{12}d_2 = d_{12}, d_{21}d_1 = d_2d_{21} = d_{21}, d_{12}d_{21} = d_1, d_{21}d_{12} = d_2, d_{12}^2 = 0 = d_{21}^2$.

The only orthogonal idempotents d_1, d_2 of R satisfying $\varphi(d_1) = c_1 + Z$, $\varphi(d_2) = c_2 + Z$ and $d_1 + d_2 = c_1 + c_2$ are $d_1 = c_1 + \mu z_{12} - \nu z_{21}$ and $d_2 = c_2 - \mu z_{12} + \nu z_{21}$ with $\mu, \nu \in k$. The Albert (= Peirce)-decomposition $R = R_{11} + R_{12} + R_{21} + R_{22}$ with respect to the complete orthogonal system of idempotents d_1, d_2 is given by

$$R_{11} = kd_1 + kz_1,$$

$$R_{12} = k(c_{12} - \mu(1 - \xi)z_1 + \nu(1 + \xi)\kappa^{-1}z_1) + kz_{12},$$

$$R_{21} = k(c_{21} + \mu(1 - \xi)\kappa z_2 - \nu(1 + \xi)z_2 - (\mu\kappa + \nu)\tau z_{12}) + kz_{21},$$

$$R_{22} = kd_2 + kz_2 \text{ [13], [14], [15].}$$

Hence the only candidates for d_{12}, d_{21} are

$$d_{12} := c_{12} - \mu(1 - \xi)z_1 + \nu(1 + \xi)\kappa^{-1}z_1 + \rho z_{12} \text{ and}$$

$$d_{21} := c_{21} + \mu(1 - \xi)\kappa z_2 - \nu(1 + \xi)z_2 - (\mu\kappa + \nu)\tau z_{12} + \sigma z_{21} \text{ with } \rho, \sigma \in k.$$

We calculate

$$d_{12}d_1 = (\rho\kappa\tau - \mu)z_1, \quad d_1d_{21} = (1 - \nu)z_2,$$

$$d_1d_{12} + d_{12}d_1 = d_{12}, \quad d_1d_{21} + d_{21}d_1 = d_{21},$$

$$d_{12}d_{21} = d_1 + (\rho\kappa\tau - \mu)z_{12} - (1 - \nu)z_{21} + (\alpha + \tau + \sigma\kappa^{-1}(1 + \xi))$$

$$+ (\mu\kappa + \nu)\tau(\xi - 1) + \kappa\tau(\mu - \rho\kappa\tau) + \tau(\nu - 1))z_1,$$

$$d_{21}d_{12} = d_2 - (\rho\kappa\tau - \mu)z_{12} + (1 - \nu)z_{21} + \kappa(\varepsilon + \rho(1 - \xi))z_2$$

$$+ (\tau(1 - \nu) + \kappa\tau(\rho\kappa\tau - \mu))z_1,$$

$$d_{21}^2 = (\varepsilon + \rho(1 - \xi))z_1,$$

$$d_{21}^2 = \kappa(\alpha + \tau + \sigma\kappa^{-1}(1 + \xi) + (\mu\kappa + \nu)\tau(\xi - 1))z_2.$$

Hence R allows a Wedderburn splitting if and only if there exist $\mu, \nu, \rho, \sigma \in k$ such that the following equations hold:

$$\nu = 1,$$

$$\mu = \rho\kappa\tau,$$

$$\rho(\xi - 1) = \varepsilon,$$

$$\sigma(\xi + 1) = -\kappa(\alpha + \tau\xi + \varepsilon\kappa^2\tau^2).$$

From this one concludes

THEOREM 1. *Let k be a field. Then $R = R(\alpha, \varepsilon, \tau, \xi, \kappa)$ does not admit a Wedderburn splitting if and only if $\xi = 1, \varepsilon \neq 0$ or $\xi = -1, \alpha \neq \tau - \varepsilon\kappa^2\tau^2$.*

Especially for $\kappa = \varepsilon = \xi = 1, \alpha = \tau = 0$ one obtains a right alternative algebra for which the Wedderburn Principal Theorem fails to hold. On the other hand for alternative algebras and some more general classes of algebras the Wedderburn Principal Theorem is true [1], [4]–[11].

Quadratic Jordan algebras. Every right alternative algebra R carries the structure of a quadratic Jordan algebra R^+ with U -operator $U_x(y) = (xy)x$ [2], [12], [13].

If 2 is an invertible element of the ground ring k , then the concepts quadratic Jordan algebras and linear Jordan algebras are equivalent [2], [12]. For finite-dimensional (linear) Jordan algebras over a field k of characteristic $\neq 2$ the Wedderburn Principal = Albert-Penico-Taft Theorem holds [2], [11]. In [11] a 4-dimensional example of a noncommutative Jordan algebra without a Wedderburn splitting was given. In case of characteristic two this example is a linear Jordan algebra not admitting a Wedderburn splitting. But in 1970 McCrimmon observed in a discussion with the author that this example admits a Wedderburn splitting as quadratic Jordan algebra since the radical has dimension three. The problem whether Jordan algebras of characteristic two admit Wedderburn splittings has resisted all efforts for many years. Now we can decide it by giving a counterexample.

THEOREM 2. *If k is a field of characteristic two then the quadratic Jordan algebra R^+ does not admit a Wedderburn splitting if and only if $\xi = 1, \varepsilon \neq 0$ or $\xi = 1, \alpha \neq \tau$.*

PROOF. The radical of R^+ is Z since $R^+/Z \cong M_2(k)^+$ is a simple quadratic Jordan algebra. If there exists a Jordan subalgebra S^+ of R^+ such that $R^+ = S^+ \oplus Z$, then there are $\mu, \nu, \rho, \sigma, \chi, \omega \in k$ such that $d_1 := c_1 + \mu z_{12} - \nu z_{21}$, $d_2 := c_2 - \mu z_{12} + \nu z_{21}$ are idempotents in S^+ and $d_{12} := c_{12} + \rho z_{12} + \sigma z_{21}$, $d_{21} := c_{21} + \chi z_{12} + \omega z_{21}$ are elements of S^+ such that $d_{12} \circ d_{21} = c_1 + c_2$. But

$$\begin{aligned} d_{12} \circ d_{21} &= c_1 + c_2 + (\alpha + \tau + \chi(1 - \xi) + \omega\kappa^{-1}(1 + \xi))z_1 \\ &\quad + (\varepsilon\kappa + \rho\kappa(1 - \xi) + \sigma(1 + \xi))z_2 \\ &\neq c_1 + c_2 \end{aligned}$$

for $\xi = 1, \varepsilon \neq 0$ or $\xi = 1, \alpha \neq \tau$. The converse follows from Theorem 1.

Isotopes of R . If R is a unital right alternative algebra and $u \in R$ is invertible, then the k -module R under the new multiplication $x \cdot_u y := (xu)y$ is a right alternative algebra $R^{(u)}$ called the u -isotope of R [12]. An algebra S isomorphic to an isotope of R is called *isotopic* to R . Since isotopy is an

equivalence relation more general than isomorphy one may ask whether for each right alternative algebra an isotope can be found admitting a Wedderburn splitting. As a first step in this direction we study the isotopes of our special class of right alternative algebras $R(\alpha, \varepsilon, \tau, \xi, \kappa)$.

An element $u \in R$ is invertible if and only if $u + Z \in R/Z$ is invertible. Hence

$$u = \lambda_1 c_1 + \lambda_2 c_2 + \lambda_{12} c_{12} + \lambda_{21} c_{21} + \rho_1 z_1 + \rho_2 z_2 + \rho_{12} z_{12} + \rho_{21} z_{21}$$

is invertible if and only if $\lambda_1 \lambda_2 - \lambda_{12} \lambda_{21} \neq 0$. Without loss of generality we may assume $\lambda_1 \lambda_2 - \lambda_{12} \lambda_{21} = 1$ (for invertible $\lambda \in k$ one has $R^{(\lambda u)} \cong R^{(u)}$). Then the inverse u^{-1} of u equals

$$u^{-1} = \lambda_2 c_1 + \lambda_1 c_2 - \lambda_{12} c_{12} - \lambda_{21} c_{21} + \sigma_1 z_1 + \sigma_2 z_2 + \sigma_{12} z_{12} + \sigma_{21} z_{21}$$

with

$$\begin{aligned}\sigma_1 &= \beta \lambda_{12} \lambda_2 - \lambda_2^2 \rho_1 - \lambda_{12}^2 \rho_2 \kappa^{-1}, \\ \sigma_2 &= \beta \lambda_{21} \lambda_1 \kappa - \lambda_1^2 \rho_2 - \lambda_{21}^2 \rho_1 \kappa, \\ \sigma_{12} &= -\rho_{12} - \beta \lambda_{12} \lambda_{21} + \lambda_2 \lambda_{21} \rho_1 + \lambda_1 \lambda_{12} \rho_2 \kappa^{-1}, \\ \sigma_{21} &= -\rho_{21} - \beta \lambda_{12} \lambda_{21} \kappa + \lambda_1 \lambda_{12} \rho_2 + \lambda_2 \lambda_{21} \rho_1 \kappa\end{aligned}$$

and β given by $\beta = (\alpha + \tau) \lambda_{21} + \varepsilon \lambda_{12} + \rho_{12}(1 - \xi) + \rho_{21} \kappa^{-1}(1 + \xi)$. The algebra $R^{(u)}$ is again unital with unit element u^{-1} . The radical of $R^{(u)}$ equals Z and satisfies $Z_u \cdot Z = 0$. The factor algebra $R^{(u)}/Z = (R/Z)^{(u+Z)}$ is isomorphic to $M_2(k)$. This leads us to consider for fixed $\mu, \nu, \rho, \sigma \in k$ the following elements of R

$$\begin{aligned}d_1 &:= \lambda_2 c_1 - \lambda_{12} c_{12} + \sigma_1 z_1 + (\sigma_{12} + \mu) z_{12} - \nu z_{21}, \\ d_2 &:= \lambda_1 c_2 - \lambda_{21} c_{21} + \sigma_2 z_2 + (\sigma_{21} + \nu) z_{21} - \mu z_{12}, \\ d_{12} &:= \lambda_1 c_{12} - \lambda_{21} c_1 + ((\sigma_{21} + \nu)(1 + \xi) \kappa^{-1} - \lambda_{21}(\alpha + \tau) - \mu(1 - \xi)) z_1 \\ &\quad + \lambda_{21} z_{21} + \rho z_{12}, \\ d_{21} &:= \lambda_2 c_{21} - \lambda_{12} c_2 + ((\sigma_{12} + \mu)(1 - \xi) \kappa - \lambda_{12} \varepsilon \kappa - \nu(1 + \xi)) z_2 \\ &\quad - (\sigma_{12} \kappa + \mu \kappa + \nu) \tau z_{12} + \sigma z_{21}.\end{aligned}$$

After calculating

$$\begin{aligned}d_1 u &= c_1 + (\lambda_{12} \lambda_{21} \tau - \lambda_1 \rho_{12} \kappa \tau + \lambda_{21} \rho_1 \kappa \tau + \mu(\lambda_1 \kappa \tau + \lambda_{12}) - \sigma_2 \lambda_{12} \tau) z_1 \\ &\quad + (\lambda_2 \lambda_{21} - \nu \lambda_{21}) z_2 + (\lambda_{21} \rho_{21} \tau - \lambda_{21} \rho_{12} \kappa \tau + (\sigma_{21} + \nu) \lambda_{21} \tau \\ &\quad \quad \quad + \mu(\lambda_2 + \lambda_{21} \kappa \tau)) \\ &\quad \cdot (z_{12} - \kappa \tau z_1) + (\lambda_{12} \lambda_{21} - \nu \lambda_1) z_{21},\end{aligned}$$

$$\begin{aligned}
d_2 u &= c_2 - (\lambda_{12}\lambda_{21}\tau - \lambda_1\rho_{12}\kappa\tau + \lambda_{21}\rho_1\kappa\tau + \mu(\lambda_1\kappa\tau + \lambda_{12}) - \sigma_2\lambda_{12}\tau)z_1 \\
&\quad - (\lambda_2\lambda_{21} - \nu\lambda_{21})z_2 - (\lambda_{21}\rho_{21}\tau - \lambda_{21}\rho_{12}\kappa\tau + (\sigma_{21} + \nu)\lambda_{21}\tau \\
&\quad \quad \quad + \mu(\lambda_2 + \lambda_{21}\kappa\tau)) \\
&\quad \cdot (z_{12} - \kappa\tau z_1) - (\lambda_{12}\lambda_{21} - \nu\lambda_1)z_{21}, \\
d_{12} u &= c_{12} + (\lambda_1\lambda_{12}\epsilon - \lambda_1\rho_{12}\xi + \lambda_1(\sigma_{21} + \rho_{21} + \nu)(1 + \xi)\kappa^{-1} - \lambda_{21}\rho_1 \\
&\quad \quad \quad - \lambda_1\lambda_{21}g\tau - \lambda_1\mu(1 - \xi) + \lambda_1\rho\kappa\tau + \lambda_{12}\rho)z_1 \\
&\quad + (\lambda_1\rho_2\kappa^{-1} - \lambda_{21}\rho_{12} - \lambda_{21}^2(\alpha + \tau) \\
&\quad \quad + \lambda_{21}(\sigma_{21} + \nu)(1 + \xi)\kappa^{-1} \\
&\quad \quad - \lambda_{21}\mu(1 - \xi) - \lambda_{21}^2\tau + \lambda_2\rho + \lambda_{21}\rho\kappa\tau)(z_{12} - \kappa\tau z_1), \\
d_{21} u &= c_{21} + (\lambda_2\lambda_{12} + \lambda_2\rho_1\kappa - \lambda_{12}\rho_{12}\kappa + \lambda_{12}(\sigma_{12} + \mu)(1 - \xi)\kappa - \lambda_{12}^2\epsilon\kappa \\
&\quad \quad \quad - \lambda_{12}\nu(\xi + 1) - (\sigma_{12}\kappa + \mu\kappa + \nu)\tau\lambda_1\kappa \\
&\quad \quad \quad - (\sigma_{12}\kappa + \mu\kappa + \nu)\lambda_{12})\tau z_1 \\
&\quad + (\lambda_2^2 - 1 + \lambda_{21}\kappa(\alpha + \tau)\lambda_2 + \lambda_2\rho_{12}(1 - \xi)\kappa + \lambda_2\rho_{21}\xi - \lambda_{12}\rho_2 \\
&\quad \quad \quad + \lambda_2(\sigma_{12} + \mu)(1 - \xi)\kappa - \lambda_2\nu(\xi + 1) + \lambda_{21}\sigma) \\
&\quad + (\lambda_2\rho_{21} - \lambda_2\rho_{12}\kappa - (\sigma_{12}\kappa + \mu\kappa + \nu)(\lambda_2 + \tau\lambda_{21}\kappa) - \lambda_{21}\sigma)\tau(z_{12} - \kappa\tau z_1) \\
&\quad + (\lambda_2\lambda_{12} + \lambda_2\rho_1\kappa - \lambda_{12}\rho_{21} + \lambda_{12}(\sigma_{12} + \mu)(1 - \xi)\kappa \\
&\quad \quad \quad - \lambda_{12}^2\epsilon\kappa - \nu\lambda_{12}(\xi + 1) + \lambda_1\sigma)z_{21}
\end{aligned}$$

one obtains

$$\begin{aligned}
d_1 u \cdot d_1 &= d_1, \quad d_2 u \cdot d_2 = d_2, \quad d_1 + d_2 = u^{-1}, \\
d_1 u \cdot d_{12} &= d_{12} + ((\lambda_{21} - \rho\kappa)\tau + \mu)z_1, \\
d_{12} u \cdot d_1 &= -((\lambda_{21} - \rho\kappa)\tau + \mu)z_1, \\
d_2 u \cdot d_{21} &= d_{21} - (\lambda_2 - \nu)z_2, \\
d_{21} u \cdot d_2 &= (\lambda_2 - \nu)z_2, \\
d_{12} u \cdot d_{12} &= ((\rho - \lambda_1\lambda_{21}\rho_{12})(1 - \xi) + \lambda_{21}(1 - \lambda_1\rho_{21})(1 + \xi)\kappa^{-1} \\
&\quad + \lambda_1(1 - \lambda_{12}\lambda_{21})\epsilon + \lambda_{21}^2\rho_1 + \lambda_1^2\rho_2\kappa^{-1} - \lambda_1\lambda_{21}^2(\alpha + \tau))z_1, \\
d_{21} u \cdot d_{21} &= (\lambda_2(1 - \lambda_{12}\lambda_{21})(\alpha + \tau)\kappa - (\sigma_{12}\kappa + \mu\kappa + \nu)(1 - \xi)\tau\kappa \\
&\quad + \sigma(\xi + 1) - \lambda_2\lambda_{12}\rho_{12}(1 - \xi)\kappa - \lambda_2\lambda_{12}\rho_{21}(1 + \xi) \\
&\quad \quad \quad + \lambda_{12}^2\rho_2 + \lambda_2^2\rho_1\kappa - \lambda_2\lambda_{12}^2\epsilon\kappa)z_2,
\end{aligned}$$

$$\begin{aligned}
d_{12} \cdot u \cdot d_{21} &= d_1 + (\lambda_2 \alpha + (\sigma_{12} \kappa + \mu \kappa + \nu) \tau (\xi - 1) \\
&\quad + \sigma(1 + \xi) \kappa^{-1} - \sigma_1 + \nu \tau) z_1 \\
&\quad + (\rho \kappa \tau - \mu - \lambda_{21} \tau)(z_{12} - \kappa \tau z_1) + (\nu - \lambda_2) z_{21}, \\
d_{21} \cdot u \cdot d_{12} &= d_2 + (\lambda_2 - \nu) \tau z_1 - (\rho \kappa \tau - \mu - \lambda_{21} \tau)(z_{12} - \kappa \tau z_1) \\
&\quad - (\nu - \lambda_2) z_{21} + (\lambda_1 \varepsilon \kappa + \lambda_{21}(1 + \xi) + \rho(1 - \xi) \kappa - \sigma_2) z_2,
\end{aligned}$$

and

$$d_{1u} \cdot z = c_1 z, \quad d_{2u} \cdot z = c_2 z, \quad z_u \cdot d_1 = z c_1, \quad z_u \cdot d_2 = z c_2 \quad \text{for all } z \in Z.$$

These equations imply that d_1, d_2 form a complete orthogonal system of idempotents of $R^{(u)}$ with Albert (= Peirce)-decomposition given by $R_{11}^{(u)} = k d_1 + k z_1$, $R_{12}^{(u)} = k d_{12} + k z_{12}$, $R_{21}^{(u)} = k d_{21} + k z_{21}$, $R_{22}^{(u)} = k d_2 + k z_2$. The canonical homomorphism $\varphi: R^{(u)} \rightarrow R^{(u)}/Z \cong M_2(k)$ maps d_1, d_2 onto an orthogonal system of idempotents of $M_2(k)$ and the Albert decomposition of $R^{(u)}$ onto the Peirce decomposition of $M_2(k)$. Assume that $R^{(u)} = S^{(u)} \oplus Z$ for some subalgebra $S^{(u)}$ of $R^{(u)}$. Then $S^{(u)} \cong M_2(k)$ is associative and contains orthogonal idempotent s_1, s_2 and elements s_{12}, s_{21} such that

$$\begin{aligned}
s_1 + s_2 &= u^{-1}, \quad s_1 s_{12} = s_{12} s_2 = s_{12}, \quad s_2 s_{21} = s_{21} s_1 = s_{21}, \\
s_{12} s_1 &= s_2 s_{12} = s_{12} s_{21} = s_{21} s_2 = s_{12}^2 = s_{21}^2 = 0
\end{aligned}$$

and $s_1 - d_1 \in Z$, $s_2 - d_2 \in Z$, $s_{12} - d_{12} \in Z$, $s_{21} - d_{21} \in Z$. Let $z := s_1 - d_1 \in Z$. Then $d_2 - s_2 = z$ and the equation $s_{1u} \cdot s_1 = d_{1u} \cdot d_1 + z_u \cdot d_1 + d_{1u} \cdot z$ implies $z \in k z_{12} + k z_{21}$. Hence, by suitable choice of $\mu, \nu \in k$ we may assume $d_1 = s_1$ and $d_2 = s_2$. From $s_{12} - d_{12} \in Z \cap R_{12}^{(u)} = k z_{12}$ and $s_{21} - d_{21} \in Z \cap R_{21}^{(u)} = k z_{21}$ we see that by suitable choice of ρ, σ we may assume $d_{12} = s_{12}$ and $d_{21} = s_{21}$. Consequently $R^{(u)}$ admits a Wedderburn splitting if and only if there exist $\mu, \nu, \rho, \sigma \in k$ such that the following equations hold

$$\begin{aligned}
\nu &= \lambda_2, \quad \mu = (\rho \kappa - \lambda_{21}) \tau, \\
(\rho - \lambda_1 \lambda_{21} \rho_{12})(1 - \xi) + \lambda_{21}(1 - \lambda_1 \rho_{21})(1 + \xi) \kappa^{-1} \\
&\quad + \lambda_1(1 - \lambda_{12} \lambda_{21}) \varepsilon + \lambda_{21}^2 \rho_1 + \lambda_1^2 \rho_2 \kappa^{-1} - \lambda_1 \lambda_{21}^2 (\alpha + \tau) = 0, \\
\lambda_2(1 - \lambda_{12} \lambda_{21})(\alpha + \tau) - (\sigma_{12} \kappa + \mu \kappa + \lambda_2)(1 - \xi) \tau + \sigma(\xi + 1) \kappa^{-1} \\
&\quad - \lambda_2 \lambda_{12} \rho_{12}(1 - \xi) - \lambda_2 \lambda_{12} \rho_{21}(1 + \xi) \kappa^{-1} \\
&\quad + \lambda_{12}^2 \rho_2 \kappa^{-1} + \lambda_2^2 \rho_1 - \lambda_2 \lambda_{12}^2 \varepsilon = 0, \\
\lambda_2 \alpha + (\sigma_{12} \kappa + \mu \kappa + \nu) \tau (\xi - 1) + \sigma(1 + \xi) \kappa^{-1} - \sigma_1 + \nu \tau &= 0, \\
\lambda_1 \varepsilon + \lambda_{21}(1 + \xi) \kappa^{-1} + \rho(1 - \xi) - \sigma_2 \kappa^{-1} &= 0.
\end{aligned}$$

In this system of linear equations for μ, ν, ρ, σ the 3rd and 4th equations are superfluous as one sees by subtracting the 6th (resp. 5th) equation and inserting the value of σ_2 (resp. σ_1). Hence $R^{(u)}$ admits a Wedderburn splitting if and only if the following equations hold

$$\begin{aligned}\nu &= \lambda_2, \mu = (\rho\kappa - \lambda_{21})\tau, \\ \rho\kappa(1 - \xi) &= \sigma_2 - \lambda_1\epsilon\kappa - \lambda_{21}(1 + \xi), \\ \sigma(1 + \xi) &= \sigma_1\kappa - \lambda_2(\alpha + \tau)\kappa - (\sigma_{12}\kappa + \mu\kappa + \nu)\tau(\xi - 1)\kappa.\end{aligned}$$

We have proved

THEOREM 3. *Let $R = R(\alpha, \epsilon, \tau, \xi, \kappa)$ and*

$$u = \lambda_1c_1 + \lambda_2c_2 + \lambda_{12}c_{12} + \lambda_{21}c_{21} + \rho_1z_1 + \rho_2z_2 + \rho_{12}z_{12} + \rho_{21}z_{21}$$

with $\lambda_1\lambda_2 - \lambda_{12}\lambda_{21} = 1$. Then the isotope $R^{(u)}$ does not admit a Wedderburn splitting if and only if

$$\xi = 1, \quad \lambda_1\epsilon\kappa + 2\lambda_{12} \neq \sigma_2 = \beta\lambda_{21}\lambda_1 - \lambda_{21}^2\kappa\rho_1 - \lambda_1^2\rho_2$$

or

$$\xi = -1, \quad \lambda_2(\alpha + \tau) - 2\tau(\sigma_{12}\kappa + \mu\kappa + \nu) \neq \sigma_1 = \beta\lambda_{12}\lambda_2 - \lambda_2^2\rho_1 - \lambda_{12}^2\kappa^{-1}\rho_2,$$

where $\beta = (\alpha + \tau)\lambda_{21} + \epsilon\lambda_{12} + \rho_{12}(1 - \xi) + \rho_{21}\kappa^{-1}(1 + \xi)$.

COROLLARY 1. *For all choices of $\alpha, \epsilon, \tau, \xi, \kappa$ ($\kappa \neq 0$) the algebra $R = R(\alpha, \epsilon, \tau, \xi, \kappa)$ has an isotope $R^{(u)}$ admitting a Wedderburn splitting.*

PROOF. If $\xi \notin \{1, -1\}$ every isotope $R^{(u)}$ of R admits a Wedderburn splitting. In case $\xi = 1 \neq -1$ or $\xi = -1 \neq 1$ the equation $\lambda_1\lambda_2 - \lambda_{12}\lambda_{21} = 1$ implies $\lambda_1 \neq 0$ or $\lambda_{21} \neq 0$ and $\lambda_2 \neq 0$ or $\lambda_{12} \neq 0$. Hence one can choose ρ_1, ρ_2 so that

$$\lambda_1^2\rho_2 + \lambda_{21}^2\kappa\rho_1 = \beta\lambda_{21}\lambda_1 - \lambda_1\epsilon\kappa - 2\lambda_{12} \text{ resp.}$$

$$\lambda_{12}^2\kappa^{-1}\rho_2 + \lambda_2^2\rho_1 = \beta\lambda_{12}\lambda_2 - \lambda_2(\alpha + \tau) + 2\tau(\sigma_{12}\kappa + \mu\kappa + \nu)$$

and $R^{(u)}$ admits a Wedderburn splitting. In the remaining case $\xi = 1 = -1$ both equations have a common solution ρ_1, ρ_2 since the determinant is $\lambda_1^2\lambda_2^2 - \lambda_{12}^2\lambda_{21}^2 = (\lambda_1\lambda_2 - \lambda_{12}\lambda_{21})^2 = 1$.

COROLLARY 2. *For all choices of $\alpha, \epsilon, \tau, \xi, \kappa$ ($\kappa \neq 0$) the quadratic Jordan algebra $R^+ = R(\alpha, \epsilon, \tau, \xi, \kappa)^+$ has an isotope $(R^+)^{(u)}$ admitting a Wedderburn splitting.*

PROOF. A Wedderburn splitting of $R^{(u)}$ induces a Wedderburn splitting of $(R^{(u)})^+ = (R^+)^{(u)}$.

Noncommutative Jordan algebras. The (left) homotopes of $A^{(u)}$ of a noncommutative Jordan algebra A are again noncommutative Jordan algebras [3]. For unital A and invertible $u \in A$ we call $A^{(u)}$ the u -isotope. Let A be the noncommutative Jordan algebra over a field k of characteristic $\neq 2$ with unit element e and basis e, x, y, z whose multiplication is determined by $x^2 = y^2 = e, xy = z = -yx, xz = zx = yz = zy = z^2 = 0$ [11]. Here one can show

THEOREM 4. *A is a noncommutative Jordan algebra such that no isotope $A^{(u)}$ of A admits a Wedderburn splitting.*

PROOF. Let $u = \rho e + \alpha x + \beta y + \gamma z \in A$. The equation $u^2 = (\alpha^2 + \beta^2 -$

$\rho^2)e + 2\rho u$ implies that u is invertible if and only if $\alpha^2 + \beta^2 - \rho^2 \neq 0$. The multiplication of $a, b \in A$ in $A^{(u)}$ is given by $a \cdot_u b := (au)b + (ub)a - u(ba)$. Hence the basis e, x, y, z of $A^{(u)}$ satisfies the multiplication table

	e	x	y	z
e	$\rho e + \alpha x + \beta y + \gamma z$	$\alpha e + \rho x - \beta z$	$\beta e + \rho y + \alpha z$	ρz
x	$\alpha e + \rho x + \beta z$	$\rho e + \alpha x - \beta y - \gamma z$	$\beta x + \alpha y + \rho z$	αz
y	$\beta e + \rho y - \alpha z$	$\beta x + \alpha y - \rho z$	$\rho e - \alpha x + \beta y - \gamma z$	βz
z	ρz	αz	βz	0

The radical is $Z = kz$. If $A^{(u)}$ admits a Wedderburn splitting, then there exist $\xi, \eta, \varepsilon \in k$ such that $x + \xi z, y + \eta z, e + \varepsilon z$ multiply in the same way as $x + Z, y + Z, e + Z$ in A/Z . The commutator in $A^{(u)}$ of a, b is $[a, b]^{(u)} = a \cdot_u b - b \cdot_u a = [u, b]a + [a, u]b + u[a, b]$. Hence $[x + \xi z, y + \eta z]^{(u)} = 2\rho z, [x + \xi z, e + \varepsilon z]^{(u)} = 2\beta z, [e + \varepsilon z, y + \eta z]^{(u)} = 2\alpha z$. There is no choice of ξ, η, ε such that $2\rho z = 2\beta z = 2\alpha z = 0$ i.e., $\alpha = \beta = \rho = 0$ since u is invertible. Hence $A^{(u)}$ does not admit a Wedderburn splitting.

ADDED IN PROOF. By letter of January 20, 1978, I. P. Shestakov informed the author that he independently constructed an eight-dimensional right alternative algebra that does not permit a Wedderburn decomposition.

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