ON ITERATED SIMILARITIES OF OPERATORS¹

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ABSTRACT. Let $\mathcal{L}(\mathcal{H})$ be the algebra of operators on a complex Hilbert space $\mathcal{H}$. If $A \in \mathcal{L}(\mathcal{H})$ is invertible and $\{X \in \mathcal{L}(\mathcal{H}): \|A^k X A^{-k}\| \leq C(X) < \infty, k = 0, 1, 2, \dots\}$ coincides with $\mathcal{L}(\mathcal{H})$, then A is a multiple of a similarity of a unitary operator.

Let $\mathcal{L}(\mathcal{H})$ be the algebra of all (bounded linear) operators acting on the complex infinite dimensional Hilbert space $\mathcal{H}$ and let A be an invertible operator. In [1], J. A. Deddens analyzed the algebra $\mathfrak{B}(A) = \{X \in \mathcal{L}(\mathcal{H}): \|A^k X A^{-k}\| \leq C(X) < \infty \text{ for } k = 0, 1, 2, \dots\}$ in connection with nest algebras, observed that if A is a multiple of a similarity of a unitary operator, then $\mathfrak{B}(A) = \mathcal{L}(\mathcal{H})$ and asked whether the converse is true. This is, indeed, the case:

Assume that $\mathfrak{B}(A) = \mathcal{L}(\mathcal{H})$ and let $c \in \sigma(A)$ (the spectrum of A). Then, by the uniform boundedness principle, there exists a constant $C > 0$ such that $\|\tau_k\| \leq C$ for all $k = 0, 1, 2, \dots$, where $\tau_k \in \text{Aut}[\mathcal{L}(\mathcal{H})]$ is the automorphism defined by $\tau_k(X) = A^k X A^{-k}$. Thus if $\mathcal{Q}$ is the von Neumann algebra $\mathcal{L}(\mathcal{H}) \oplus \mathcal{L}(\mathcal{H}) \oplus \dots$ (denumerably many copies of $\mathcal{L}(\mathcal{H})$), then $\tau(X_0, X_1, X_2, \dots, X_k, \dots) = (X_0, A X_1 A^{-1}, A^2 X_2 A^{-2}, \dots, A^k X_k A^{-k}, \dots) = (\tau_0(X_0), \tau_1(X_1), \dots, \tau_k(X_k), \dots)$ defines an automorphism of $\mathcal{Q}$ that fixes every summand of $\mathcal{Q}$ (i.e., $\tau(0, 0, \dots, 0, X \text{ (} k\text{th coordinate), } 0, \dots) = (0, 0, \dots, 0, \tau_k(X), 0, \dots)$ for all $k = 0, 1, 2, \dots$). Combining this observation with [2, Corollary 4.1.22], we see that τ is inner, i.e., there exists $T = (T_0, T_1, T_2, \dots, T_k, \dots)$ invertible in $\mathcal{Q}$ such that if $X = (X_0, X_1, X_2, \dots, X_k, \dots) \in \mathcal{Q}$, then

$$\tau(X) = T X T^{-1} = (T_0 X_0 T_0^{-1}, T_1 X_1 T_1^{-1}, T_2 X_2 T_2^{-1}, \dots, T_k X_k T_k^{-1}, \dots).$$

Comparing this expression of τ with its definition, it readily follows that $T_k = c_k A^k$ for suitably chosen constants c_k , $k = 0, 1, 2, \dots$. Since T is invertible, $\sigma(T) \subset \{z: a = \|T^{-1}\|^{-1} \leq |z| \leq \|T\| = b\}$, whence we conclude that $a \leq |c_k c^k| \leq b$, because $\sigma(T) \supset \bigcup_{k=0}^{\infty} \sigma(T_k) = \bigcup_{k=0}^{\infty} c_k \sigma(A^k)$.

Hence, $\|[(1/c)A]^k\| \leq b/a$ and $\|[(1/c)A]^{-k}\| \leq b/a$ for all $k = 0, 1, 2, \dots$. By a well-known theorem of B. Sz. Nagy [3], it follows that $A = c W U W^{-1}$, where W is invertible and U is unitary.

The author wishes to thank the referee for the following observation: "ad hoc" modifications of the same proof yield a stronger result. Namely,

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THEOREM. *If $\mathfrak{B}(A)$ is a C^* -algebra and every automorphism of $\mathfrak{B}(A)$ has the form $\tau(X) = TXT^{-1}$ ($X \in \mathfrak{B}(A)$) for some invertible $T \in \mathcal{L}(\mathcal{H})$, then A is a multiple of a similarity of a unitary operator and $\mathfrak{B}(A) = \mathcal{L}(\mathcal{H})$.*

The referee also remarked that J. Stampfli has independently proved that A is a multiple of a similarity of a unitary operator whenever $\mathfrak{B}(A)$ contains the compact operators.

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