REPRESENTING A CLOSED OPERATOR AS A QUOTIENT OF CONTINUOUS OPERATORS

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ABSTRACT. The closed operators in a Hilbert space H are characterized as quotients AB^{-1} of continuous operators on H such that the vector sum $A^*(H) + B^*(H)$ is closed. This leads to the function $\Gamma(A) = A(1 - A^*A)^{-1/2}$, which is shown to map the strictly contractive operators on H reversibly onto the closed densely-defined operators, so as to preserve the selfadjoint and nonnegative conditions.

Introduction. Suppose that $(H, \langle \cdot, \cdot \rangle)$ is a complete complex inner product space (e.g. a Hilbert space) with corresponding norm $\|\cdot\|$ such that for each x in H , $\|x\| = \langle x, x \rangle^{1/2}$, and let $(H \times H, \langle \cdot, \cdot \rangle')$ denote the usual product space with corresponding norm $\|\cdot\|'$. By a *closed operator* in H is meant a linear function from a linear subspace of H into H which is closed in $H \times H$. Let $\mathfrak{B}(H)$ denote the algebra of all continuous linear functions from H into H ("including" the complex numbers) and for each A in $\mathfrak{B}(H)$, let A^* denote the adjoint of A , let A^{-1} denote the inverse of the restriction of A to the closure of $A^*(H)$, and—in case A is nonnegative—let $A^{-1/2}$ denote $(A^{1/2})^{-1}$. Note that for each A in $\mathfrak{B}(H)$, $A^{-1}A$ is the orthogonal projection from H onto the closure of $A^*(H)$.

THEOREM 1. *If C is a subset of $H \times H$ then the following two statements are equivalent:*

- (1) *C is a closed operator in H , and*
- (2) *C is the quotient AB^{-1} of a member (A, B) of $\mathfrak{B}(H) \times \mathfrak{B}(H)$ such that the vector sum $A^*(H) + B^*(H)$ is closed in H .*

PROOF. Suppose (1) is true. Let S denote the domain of C and for all (x, y) in $S \times S$, let

$$\langle x, y \rangle' = \langle (x, Cx), (y, Cy) \rangle'.$$

Since C is linear and closed, $(S, \langle \cdot, \cdot \rangle')$ is a complete inner product space; moreover, for each x in S ,

$$\|x\|^2 \leq \langle x, x \rangle' \quad \text{and} \quad \|Cx\|^2 \leq \langle x, x \rangle'.$$

It follows from a result of Mac Nerney [3, Theorem 3, p. 666] that there is a nonnegative member B of $\mathfrak{B}(H)$ such that $B(H) = S$ and for each (x, y) in

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$S \times S$, $\langle x, y \rangle' = \langle B^{-1}x, B^{-1}y \rangle$. (An argument appears in the proof in [4, p. 110].) Let $A = CB$: if x is in H then

$$\|Ax\|^2 \leq \langle Bx, Bx \rangle' \leq \|x\|^2.$$

Thus A is in $\mathfrak{B}(H)$. Now let P denote the orthogonal projection $B^{-1}B$. Then $AB^{-1} = C$, $AP = A$, and for all (x, y) in $H \times H$,

$$\langle x, [B^2 + A^*A]y \rangle = \langle (Bx, CBx), (By, CBy) \rangle' = \langle Bx, By \rangle' = \langle x, Py \rangle.$$

Hence $B^2 + A^*A = P$, and since $A^*(H) + B(H) = [B^2 + A^*A]^{1/2}(H)$ [1, Theorem 2.2, p. 260], it follows that $A^*(H) + B(H)$ is closed. Therefore (2) is true.

Now suppose (2), and let $P = B^{-1}B$ as before. We first show that C is also the quotient $(AP)B^{-1}$, and that $(AP)^*(H) + B^*(H)$ is also closed—so that it suffices to consider only the case in which $AP = A$. Clearly $C = (AP)B^{-1}$, since BB^{-1} is simply the identity function on $B(H)$. Suppose z is a member of the closure of $(AP)^*(H) + B^*(H)$. The fact that $(AP)^* = PA^*$ implies that z is in the closure of $B^*(H)$ and therefore in the closed set $A^*(H) + B^*(H)$. Thus there is a member (x, y) of $A^*(H) \times B^*(H)$ for which $x + y = z$. Since each of y and z is in the closure of $B^*(H)$, so is x . Hence $Px = x$, so that x is in $PA^*(H)$ and therefore z is in $(AP)^*(H) + B^*(H)$; the latter set must therefore be identical with its closure.

Suppose now that $AP = A$. Note that in this case, the function whose value at each x in H is (Bx, Ax) —which maps the closure of $B^*(H)$ reversibly onto C —also maps H onto C . Let D denote $[A^*A + B^*B]^{1/2}$ and consider the linear function J defined on the closed set $A^*(H) + B^*(H)$ by

$$J(z) = (BD^{-1}z, AD^{-1}z), \quad \text{for all } z \text{ in } D(H).$$

Since $A^*(H) + B^*(H)$ includes the closure of $B^*(H)$, J maps the former onto C . The following calculation for each z in $D(H)$ shows that J is an isometry from $A^*(H) + B^*(H)$ onto C :

$$\|Jz\|^2 = \|BD^{-1}z\|^2 + \|AD^{-1}z\|^2 = \langle D^{-1}z, [B^*B + A^*A]D^{-1}z \rangle = \|z\|^2.$$

Thus C is closed—i.e., (1) holds true. This completes the proof.

The first half of the proof provides a construction of such a pair (A, B) as in (2) with certain additional properties, as indicated in the following:

COROLLARY. *If C is a closed operator in H then C is the quotient AB^{-1} of a pair (A, B) of $\mathfrak{B}(H) \times \mathfrak{B}(H)$ such that B is nonnegative, $AB^{-1}B = A$, C has domain $B(H)$ and range $A(H)$, and $A^*A + B^2$ is the orthogonal projection $B^{-1}B$ onto the closure of $B(H)$ —which closure is the set $A^*(H) + B(H)$.*

REMARK. If $C = AB^{-1}$ as in the preceding corollary then the orthogonal projections from $H \times H$ onto the subsets C and C^* thereof, respectively, are easily computed as

$$\begin{bmatrix} B^2 & BA^* \\ AB & AA^* \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 - AA^* & AB \\ BA^* & 1 - B^2 \end{bmatrix}.$$

The formula for the projection onto C is a slight extension of that given by von Neumann in [7, Appendix II, pp. 72–73] for the case in which C has domain dense in H —so that C^* is a function and the selfadjoint operator $1 + C^*C$ is available. These connections are elaborated upon in Theorem 2.

NOTATION. Let $\mathcal{V}(H)$ denote the subset of $\mathcal{B}(H)$ to which A belongs only in case A is a *contraction* (i.e., $\|Ax\| < \|x\|$ for all nonzero x in H), and let $\mathcal{C}(H)$ denote the set of all closed operators in H with domain dense in H .

THEOREM 2. *The equations*

$$\Gamma(A) = A(1 - A^*A)^{-1/2}, \quad \text{for all } A \text{ in } \mathcal{V}(H),$$

define a reversible function Γ from $\mathcal{V}(H)$ onto $\mathcal{C}(H)$ with inverse function defined by

$$\Gamma^{-1}(C) = C[(1 + C^*C)^{-1}]^{1/2}, \quad \text{for all } C \text{ in } \mathcal{C}(H).$$

Moreover, Γ maps certain subsets of $\mathcal{V}(H)$ into $\mathcal{C}(H)$ as follows:

- (i) *the set of all members of $\mathcal{V}(H)$ with norm less than 1 is mapped onto $\mathcal{B}(H)$,*
- (ii) *the set of all Hermitian (symmetric) members of $\mathcal{V}(H)$ is mapped onto the set of all selfadjoint operators in H , and*
- (iii) *the set of all nonnegative members of $\mathcal{V}(H)$ is mapped onto the set of all nonnegative selfadjoint operators in H .*

PROOF. Suppose A is in $\mathcal{V}(H)$ and let B denote the positive member $(1 - A^*A)^{1/2}$ of $\mathcal{V}(H)$. Then $\Gamma(A) = AB^{-1}$ and $A^*A + B^2 = 1$; therefore by Theorem 1, $\Gamma(A)$ is a closed operator in H . Since the domain of $\Gamma(A)$ is $B(H)$ —which is dense in H (B is positive), $\Gamma(A)$ belongs to $\mathcal{C}(H)$.

Now suppose C is in $\mathcal{C}(H)$. By the corollary to Theorem 1, there is an (A, B) in $\mathcal{B}(H) \times \mathcal{B}(H)$ such that $C = AB^{-1}$, B is positive (B has range the domain of C), and $A^*A + B^2 = 1$ —i.e., $B = (1 - A^*A)^{1/2}$. For each nonzero x in H , $\|x\|^2 - \|Ax\|^2 = \|Bx\|^2 > 0$; thus A is in $\mathcal{V}(H)$ and $\Gamma(A) = C$. Now C is the set of all (Bz, Az) for z in H , so C^* is all (x, y) in $H \times H$ such that $\langle x, Az \rangle = \langle y, Bz \rangle$ for all z in H —equivalently, such that $\langle A^*x, z \rangle = \langle By, z \rangle$ for all z in H . Thus $C^* = B^{-1}A^*$. Von Neumann has shown [6, p. 301] that $1 + C^*C$ is reversible with inverse in $\mathcal{B}(H)$ (see also [5, p. 307]), and for each x in H we have that

$$\begin{aligned} x &= B^2x + B^{-1}(1 - B^2)B^{-1}B^2x \\ &= (1 + B^{-1}A^*AB^{-1})B^2x = (1 + C^*C)B^2x. \end{aligned}$$

Hence $(1 + C^*C)^{-1} = B^2$, and therefore $A = CB = C[(1 + C^*C)^{-1}]^{1/2}$. This establishes that Γ is reversible from $\mathcal{V}(H)$ onto $\mathcal{C}(H)$, with inverse as indicated.

To see that (i) is true, observe that for each A in $\mathcal{V}(H)$, there is a positive number m such that $1 - A^*A \gg m$ only in case $(1 - A^*A)^{1/2}$ is invertible [5, p. 266], and that this latter holds true provided that $\Gamma(A)$ has domain all of H .

By the closed graph theorem, this is equivalent to $\Gamma(A)$ belonging to $\mathfrak{B}(H)$.

The remainder of the proof involves the orthogonal projections described in the preceding remark. First, suppose that A is in $\mathfrak{V}(H)$; let $B = (1 - A^*A)^{1/2}$ and consider the associated projections from $H \times H$ onto $\Gamma(A)$ and $\Gamma(A)^*$. Clearly $\Gamma(A) = \Gamma(A)^*$ only in case $B^2 = 1 - AA^*$ and $AB = BA^*$. Since $A^*A + B^2 = 1$, this is equivalent to: A is normal (so that A commutes with B^2 and thus with B) and $AB = A^*B$. Since $B(H)$ is dense in H , the latter condition alone is in fact equivalent to $A = A^*$. Hence (ii) is true.

Finally, suppose A is an Hermitian member of $\mathfrak{V}(H)$, and let B denote $(1 - A^2)^{1/2}$. Then A commutes with $B^{1/2}$, so for each x in H ,

$$\langle B^{1/2}x, AB^{1/2}x \rangle = \langle Bx, Ax \rangle = \langle Bx, CBx \rangle.$$

Since $B^{1/2}(H)$ is dense in H , A is nonnegative only in case $\langle z, Az \rangle \geq 0$ for all z in $B^{1/2}(H)$. On the other hand, C is nonnegative only in case $\langle y, Cy \rangle \geq 0$ for all y in $B(H)$. Thus the preceding equalities imply the equivalence of these two conditions. Therefore (iii) is true.

FURTHER REMARKS. Theorem 2 may be regarded as a refinement of a result of J. J. Koliha, who proves in [2, pp. 778–779] that a member of $\mathfrak{B}(H)$ has norm less than one only in case it is $C(1 + C^*C)^{-1/2}$ for some C in $\mathfrak{B}(H)$. Identification of the function Γ now affords us a “natural” topology for $\mathcal{C}(H)$ with respect to which $\mathfrak{B}(H)$ is an open set dense therein. Additionally, one may derive therefrom that the vector sum of the domain and range of any selfadjoint operator in H is all of H (via the corollary to Theorem 1).

In conclusion, there is some sharpening of the Cayley transform [5, p. 325]. Let $C = \Gamma(A)$ and $B = (1 - A^*A)^{1/2}$ as usual. It is easily verified that C is Hermitian only in case $A^*B = BA$. In any case, $(C - i)(C + i)^{-1} = (A - iB)(A + iB)^{-1}$ —thus the usual representation of the Cayley transform as a quotient of possibly discontinuous operators may be replaced by one such as in Theorem 1. Moreover, C is Hermitian only in case both $A + iB$ and $A - iB$ are linear isometries from H into itself, inasmuch as

$$\|(A \pm iB)x\|^2 = \|x\|^2 \pm i\langle x, (BA - A^*B)x \rangle,$$

for all x in H .

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