TOPOLOGICAL MIXING OF HIGHER DEGREES

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ABSTRACT. We give examples of homeomorphisms which are topologically
1-mixing but not topologically 2-mixing. One is a subshift and the other is a
diffeomorphism of the torus.

Introduction. The question of whether measure-theoretic mixing implies
mixing of all degrees has been an open problem in ergodic theory for many
years [1]. This note deals with the analogous notions (which are much cruder)
in topological dynamics.

Let $T: X \to X$ be a homeomorphism of a compact metric space. One says
that T is 1-mixing if: for all nonempty open sets U, V, there exists M such
that if $m \geqslant M$ then $T^m(U) \cap V \neq \varnothing$. One says that T is 2-mixing if: for all
nonempty open sets U, V, and W, there exists M such that if $m_1, m_2 \geqslant M$
then $T^{m_2}(T^{m_1}(U) \cap V) \cap W \neq \varnothing$. For continuous one-parameter flows, one
has the obvious analogous definitions.

It is clear that 2-mixing implies 1-mixing. In this note, we give two very
simple examples which both show that 1-mixing does not imply 2-mixing.
This has also been done by M. Dekking and M. Keane [3]. Their example is a
substitution minimal set, while both of our examples have a fixed point. None
of these examples is measure-theoretically mixing.

Our first example is a subshift (i.e., a symbolic dynamical system). It has
exactly two ergodic invariant measures. One is the point mass at the fixed
point and the other is a (measure-theoretically) weakly mixing, but not
strongly mixing, measure which is positive on open sets.

Our second example is a continuous 1-parameter flow (a Stepanoff flow [2])
on the 2-dimensional torus. The orbits of this flow are the same as the orbits
of an irrational flow except that exactly one orbit of the latter splits into three
orbits of the former: a fixed point and two half-lines which die at the fixed
point, one in forward time and the other in backward time. Since the flow will
be 1-mixing but not 2-mixing, it is easy to see that the same holds for every
nonzero time t map. Thus, we get a homeomorphism of the torus which is
1-mixing but not 2-mixing. With a more careful analysis, one can make the
flow smooth and, hence, get a diffeomorphism of the torus which is 1-mixing
but not 2-mixing.

We mention that while both our examples have zero entropy, one can make positive entropy examples by simply taking the direct product of one of our maps with any positive entropy, 1-mixing map (e.g., the full shift on two symbols).

Finally, the example in §1 can be generalized to give examples of n-mixing maps which are not $(n + 1)$-mixing (with the obvious definitions). Since this paper was written, B. Weiss has found even simpler symbolic examples.

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1. The symbolic example. Let $\{0, 1\}$ denote the two-point space with the discrete topology. Let $\Sigma = \{0, 1\}^Z$. A point $x \in \Sigma$ is a bi-infinite sequence of 0's and 1's, $x = \cdots x_{-2}x_{-1}x_0x_1x_2 \cdots$. The shift homeomorphism $T: \Sigma \to \Sigma$ is defined by

$$T(x) = y \quad \text{where } y_i = x_{i+1}.$$

If $\Lambda \subset \Sigma$ is a closed T-invariant subset, then $T|_\Lambda$ is called a subshift. One way to construct a subshift is to take the T-orbit-closure of a point $x \in \Sigma$. The following characterizes mixing properties of these subshifts in terms of the blocks $(c_1 \cdots c_n \in \{0, 1\}^n)$ which appear in x.

PROPOSITION 1. *Let Λ be the T-orbit-closure of* x. *Then*

(a) $T|_\Lambda$ *is* 1-*mixing if and only if: for all finite blocks $c_1 \cdots c_n$ which appear in* x, *there exists M such that if $m \geqslant M$ then one can find $i \in Z$ such that*

$$x_i = c_1, \quad x_{i+1} = c_2, \ldots, \quad x_{i+n-1} = c_n$$

and

$$x_{i+m} = c_1, \quad x_{i+m+1} = c_2, \ldots, \quad x_{i+m+n-1} = c_n.$$

(b) $T|_\Lambda$ *is* 2-*mixing if and only if: for all finite blocks $c_1 \cdots c_n$ which appear in* x, *there exists M such that if $m_1, m_2 \geqslant M$ then one can find $i \in Z$ such that*

$$x_i = c_1, \quad x_{i+1} = c_2, \ldots, \quad x_{i+n-1} = c_n,$$
$$x_{i+m_1} = c_1, \quad x_{i+m_1+1} = c_2, \ldots, \quad x_{i+m_1+n-1} = c_n$$

and

$$x_{i+m_1+m_2} = c_1, \quad x_{i+m_1+m_2+1} = c_2, \ldots, \quad x_{i+m_1+m_2+n-1} = c_n.$$

The proof of this proposition is straightforward and well known; it follows from the fact that each block represents an open set and the collection of these open sets forms a basis for the topology.

Let $a_0 = 0$ and, inductively, $a_n = a_{n-1}b_na_{n-1}$ (juxtaposition), where b_n is the block consisting of n consecutive 1's. Observe that if $k > n$, then a_n is an initial block of a_k. Thus, it makes sense to define x^+ as the one-sided infinite sequence of 0's and 1's which has each a_n as an initial block. Define x by

$$x_i = \begin{cases} x_i^+ & \text{if } i \geqslant 0, \\ 1 & \text{if } i < 0. \end{cases}$$

PROPOSITION 2. *Let Λ be the T-orbit-closure of* x. *Then $T|_\Lambda$ is* 1-*mixing but not* 2-*mixing.*

PROOF. First observe that for every n we can write the point $\mathbf{x}^+$ as follows:

$$\mathbf{x}^+ = a_{n-1}b_{i_1}a_{n-1}b_{i_2}a_{n-1}b_{i_3}\cdots, \qquad (*)$$

where

$$i_j = n \quad \text{if } j \text{ is odd,}$$

$$i_j > n \quad \text{if } j \text{ is even.}$$

Moreover, $\{i_j\}_j \supset [n, +\infty)$.

This implies that Proposition 1(a) holds for every block which is contained in some a_{n-1}. But this includes every block which occurs in $\mathbf{x}$. So, $T|_\Lambda$ is 1-mixing.

We show that $T|_\Lambda$ is not 2-mixing for the a_0-block. To do this, it suffices to show that there is an infinite sequence $k_n \to \infty$ such that in $\mathbf{x}$ one never sees a block of the form

$$x_i x_{i+1} \cdots x_{i+k_n} \cdots x_{i+2k_n}, \qquad (**)$$

where $x_i = x_{i+k_n} = x_{i+2k_n} = 0$. We claim that if $k_n = \text{length}(a_{n-1}) + \text{length}(b_n) - 1$, then $\{k_n\}$ is such a sequence.

Suppose that a block of the form $(**)$ appears in $\mathbf{x}$. We seek a contradiction. Since there are no zeros in the b-blocks, x_i must occur in one of the a_{n-1} blocks in $(*)$ above. But then by the choice of k_n, it is easy to see that x_{i+k_n} must appear in the next a_{n-1} block to the right. And x_{i+2k_n} must appear in the next a_{n-1} block after that. But then one of the intervening blocks b_{i_j} must be b_n. For definiteness, let us say that it is the b block in between x_i and x_{i+k_n}. (The other case may be treated in exactly the same manner.) But then it is not hard to see, from the definition of k_n, that $x_{i+k_n} = x_{i-1}$. Thus, both x_{i-1} and x_i are zero. But from the construction of the sequence $\mathbf{x}$, it is evident that there are never two zeros in a row.

2. The Stepanoff flow. We shall now give an example of a continuous flow on the torus T^2 which is 1-mixing, but not 2-mixing. The example is described in the introduction and is called a Stepanoff flow. Oxtoby observed [2] that since the two half-lines (involved in the definition) are dense, every continuous parametrization of this flow is 1-mixing.[3] We shall produce a continuous parametrization which is not 2-mixing. We start with just the orbits of a Stepanoff flow, and will inductively define a parametrization.

Fix a flow box $F = I \times I$ (for the original irrational flow) which contains the fixed point $(0, 0)$ of the Stepanoff flow. (Note: $I = [-1, 1]$.) We assume that the flow moves from left to right. Fix a closed disk $A \subset F^c$. Let $\varepsilon_n \downarrow 0$ and $F_n = I \times (-\varepsilon_n, \varepsilon_n)$. We assume that outside F_1, the flow is defined by the unit speed parametrization. Now we describe how to define the parametrization on $F_n - F_{n+1}$ assuming that it has been defined outside F_n. The condition will essentially involve only the "crossing time" function (i.e., the function $f: I \to (0, +\infty]$ which tells how long it takes to cross $I \times I$).

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[3]Simply observe that every open set meets the half-line which approaches the fixed point and thus gets stretched out along the half-line which emanates from the fixed point.

For each $x \in T^2 - \text{int}(F_n)$, there is a smallest time $s(x) \geqslant 0$ such that $\varphi_{s(x)}(x) \in \{-1\} \times (-\varepsilon_n, \varepsilon_n)$. This makes sense since the parametrization is defined outside F_n. Let $S_n = \max_{x \in A}\{s(x)\}$. Note $S_n < \infty$ since $s(x)$ is a semicontinuous function on a compact set.

Choose $\delta_n \in (0, \varepsilon_n)$ so that if $\gamma \in (-\varepsilon_n, -\delta_n) \cup (\delta_n, \varepsilon_n)$ and $x = (1, \gamma)$, then

$$\varphi_{s(x)}(x) \in \{-1\} \times \{-\delta_n, \delta_n\}.$$

Assume also that $\delta_n > \varepsilon_{n+1}$.

Extend the parametrization to $F_n - F_{n+1}$ so that the crossing time function is monotonic on each side of zero and is at least S_n on $(I \times [-\delta_n, \delta_n]) \cap F_{n+1}^c$.

The induction process gives a continuous parametrization on $T^2 - (I \times \{0\})$. It is easy to see that with a little care[4] (in the induction process) this parametrization extends continuously to all of T^2 in such a way that the resulting flow is a Stepanoff flow. The important point is that with such a parametrization the crossing time across each $I \times [-\delta_n, \delta_n]$ is at least S_n. Note also that there are many such parametrizations.

In fact, Jenny Harrison has shown that flows with this type of singularity are smoothable–in the sense that there is a homeomorphism h such that the flow $h \circ \phi_s \circ h^{-1}$ is smooth.

PROPOSITION 3. *Given any Stepanoff flow $\{\varphi_s\}$ parametrized as described, $\{\varphi_s\}$ is not 2-mixing.*

PROOF. Let A, F_n be as above. We claim that $\varphi_{S_n}(\varphi_{S_n}(A) \cap A) \subset F_n$, hence $\varphi_{S_n}(\varphi_{S_n}(A) \cap A) \cap A = \varnothing$. Observe $S_n \to \infty$. Clearly, this would prove the theorem.

Suppose $y \in A$ and $\varphi_{S_n}(y) \in A$. By time S_n, the positive semiorbit $\mathcal{O}$ from y has entered F_n. But since it has returned to A by this time, it must have entered F_n along $[\{-1\} \times (\delta_n, \varepsilon_n)] \cup [\{-1\} \times (-\varepsilon_n, -\delta_n)]$ where the crossing time can be less than S_n.

Let $\varphi_{t_1}(y)$ denote the first exit point (from F_n) for $\mathcal{O}$, and let $\varphi_{t_2}(y)$ denote the next entrance point (into F_n). (See Figure 1.)
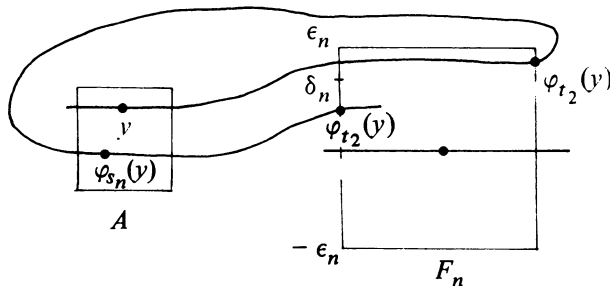

FIGURE 1

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[4]Clearly the crossing times must approach ∞ as n does.

By the choice of δ_n, $\varphi_{t_2}(y) \in \{-1\} \times (-\delta_n, \delta_n)$. Since $\varphi_{S_n}(y) \in A$ and hence is not in F_n, one sees that (as indicated in Figure 1) $t_1 < S_n < t_2$. Thus, by the definition of S_n (applied to $x = \varphi_{S_n}(y)$) and the fact that the minimum crossing time in $\{-1\} \times (-\delta_n, \delta_n)$ is at least S_n, we have $\varphi_{2S_n}(y) \in F_n$ as desired.

References

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