ON THE EXISTENCE OF DENSE ANALYTIC SUBGROUPS

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ABSTRACT. The purpose of this paper is to establish a necessary and sufficient condition for a connected Lie group to contain a proper dense connected Lie subgroup.

1. Introduction. By an analytic group and an analytic subgroup of a Lie group we mean a connected Lie group and a connected Lie subgroup, respectively. An analytic subgroup of $GL(n, \mathbf{R})$ will be referred to as a linear analytic group. For an analytic group G , $Z(G)$ and $Z_0(G)$ will denote the center of G and the identity component group of the center of G , respectively, and $\mathfrak{L}(G)$ will denote the Lie algebra of G . G' will denote the commutator subgroup of G , so that, for example, if $K(G)$ is a maximal compact subgroup of G , then $K'(G)$ will denote the commutator subgroup of $K(G)$.

The purpose of this paper is to establish a necessary and sufficient condition for an analytic group to contain a proper dense analytic subgroup. Several authors have elucidated the relationship between an analytic group and its dense analytic subgroups (cf. Djoković [1], Goto [2] and [3], and Zerling [6]) and in [5] Rajagopalan and Subrahmanian have developed a necessary and sufficient condition that a nondiscrete locally compact abelian group contain a proper dense subgroup.

2. Main results.

LEMMA 1. *Let G be an analytic group and let N be a closed normal analytic subgroup of G such that G/N is a vector group W of dimension at least one. Then $G = N \cdot V_k \cdot V_{k-1} \cdots V_1$, where V_i , $1 \leq i \leq k$, are closed one dimensional vector subgroups of G such that $N \cap V_k = N \cdot V_k \cdots V_{i+1} \cap V_i = \{e\}$, $1 \leq i \leq k-1$.*

PROOF. Let $\pi: G \rightarrow G/N$ be the natural projection. Let $W = W_k \cdot W_{k-1} \cdots W_1$ be a direct product decomposition of W into one dimensional vector subgroups. Then $\pi^{-1}(W_i) = NV_i$, where V_i is a closed one dimensional vector subgroup of G and $N \cap V_i = \{e\}$. Since N is normal in G , we have $G = \pi^{-1}(W) = N \cdot V_k \cdot V_{k-1} \cdots V_1$. Since N is the kernel of π and since $W = W_k \cdot W_{k-1} \cdots W_1$ is a direct product decomposition, we see that $N \cap V_k = N \cdot V_k \cdots V_{i+1} \cap V_i = \{e\}$, $1 \leq i \leq k-1$.

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LEMMA 2. *Let L be an analytic group and let $K(L)$ be a maximal compact subgroup of L . Suppose that L' is closed in L . Then the following statements are equivalent:*

(a) $Z_0(K(L))$ contains a two dimensional toral subgroup whose intersection with L' is finite.

(b) $\dim K(L) - \dim K(L') \geq 2$, where $K(L')$ is a maximal compact subgroup of L' .

PROOF. Since L' is closed and normal in L , we see from Iwasawa [4] that $K(L) \cap L'$ is a maximal compact subgroup of L' , and since any two maximal compact subgroups of an analytic group are conjugate, we may assume that $K(L') = K(L) \cap L'$. We have $K(L) = K'(L) \cdot Z_0(K(L))$, where $K'(L) \cap Z_0(K(L))$ is finite, and

$$\mathcal{L}(K(L)) = \mathcal{L}(K'(L)) + \mathcal{L}(Z_0(K(L)))$$

is a direct sum. Since $K'(L)$ is contained in $K(L')$, if we let $\mathcal{Q}$ denote the projection of $\mathcal{L}(K(L'))$ into $\mathcal{L}(Z_0(K(L)))$ we have the direct sum

$$\mathcal{L}(K(L')) = \mathcal{L}(K'(L)) + \mathcal{Q}.$$

Let T_1 denote the toral subgroup of $Z_0(K(L))$ generated by $\mathcal{Q}$ and let T be a toral subgroup of $Z_0(K(L))$ so that $Z_0(K(L)) = T_1 \cdot T$, $T_1 \cap T = \{e\}$. Then $K(L') = K'(L) \cdot T_1$, and $K(L) = K(L') \cdot T$, where $K(L') \cap T$ is finite. Therefore, $\dim K(L) - \dim K(L') \geq 2$ if and only if $\dim T \geq 2$ and our proof is now complete.

THEOREM. *Let L be an analytic group and let $K(L)$ be a maximal compact subgroup of L . Then in order that L contain a proper dense analytic subgroup it is necessary and sufficient that one of the following conditions be satisfied:*

(a) L' is closed in L and $\dim K(L) - \dim K(L') \geq 2$, where $K(L')$ is a maximal compact subgroup of L .

(b) L' is not closed in L .

PROOF. (i) To prove sufficiency let us first suppose that (a) is true. Then from Lemma 2 we have that $Z_0(K(L))$ contains a two dimensional toral subgroup whose intersection with L' is finite. Let N be a maximal closed normal analytic subgroup of L such that N contains L' and such that $Z_0(K(L))$ contains a two dimensional toral subgroup whose intersection with N is finite.

We have $K(L) = K'(L) \cdot Z_0(K(L))$, where $K'(L) \cap Z_0(K(L))$ is finite. Let T_1 denote the identity component group of $Z_0(K(L)) \cap N$. Then there is a toral subgroup T of $Z_0(K(L))$ such that the dimension of T is at least two and $Z_0(K(L)) = T_1 \cdot T$, $T_1 \cap T = \{e\}$. We see that NT is a closed normal analytic subgroup of L containing both $K(L)$ and L' , and $N \cap T$ is finite.

If $L = NT$, then letting V be a dense one dimensional vector subgroup of T we see that $G = NV$ is a proper dense analytic subgroup of L . We now show that $L = NT$.

If $L \neq NT$, then since NT contains $K(L)$ and L' we have that L/NT is a vector group of dimension at least one. Then by Lemma 1 we have $L = NTV_k \cdot V_{k-1} \cdots V_1$, where V_i , $1 \leq i \leq k$, are closed one dimensional vector subgroups of L such that $NT \cap V_k = NTV_k \cdots V_{i+1} \cap V_i = \{e\}$, $1 \leq i \leq k-1$. NV_k is a closed normal analytic subgroup of L properly containing N . Also $NV_k \cap T$ is finite since $NT \cap V_k = \{e\}$ and $N \cap T$ is finite. This contradicts the maximality of N . Hence $L = NT$, and so L contains a proper dense analytic subgroup.

Suppose now that (b) is true, that is, L' is not closed in L . Then from Goto [2, Proposition 14] we see that $K(L)$ is not contained in L' . Let N be a maximal analytic subgroup of L containing L' , but not containing $K(L)$.

As a first case suppose that $N \neq \bar{N}$. If $L \neq \bar{N}$, then since $\bar{N}$ contains $K(L)$ by the maximality of N we see that $L/\bar{N}$ is a vector group of dimension at least one. Therefore, from Lemma 1 there exist closed one dimensional vector subgroups $V_1, V_2, V_3, \dots, V_k$ of L so that $L = \bar{N} \cdot V_k \cdot V_{k-1} \cdots V_1$, where $\bar{N} \cap V_k = N \cdot V_k \cdots V_{i+1} \cap V_i = \{e\}$, $1 \leq i \leq k-1$. Then $N \cdot V_k$ properly contains N , but NV_k does not contain $K(L)$, since N is closed in NV_k and N is not closed in L . This contradicts the maximality of N . Therefore, $L = \bar{N}$, and N is our desired proper dense analytic subgroup.

As a second case suppose $N = \bar{N}$. We have $K(L) = K'(L) \cdot Z_0(K(L))$, where $K'(L) \cap Z_0(K(L))$ is finite. Let T_1 denote the identity component group of $Z_0(K(L)) \cap N$. Since $K'(L)$ is contained in N , but $K(L)$ is not contained in N , there is a toral group T of dimension at least one so that $Z_0(K(L)) = T_1 \cdot T$, $T_1 \cap T = \{e\}$, and NT is a closed normal analytic subgroup of L with $N \cap T$ finite. Since NT contains both L' and $K(L)$ (by the maximality of N), L/NT is a vector group. Using Lemma 1 and the maximality of N we can see as before that $L = NT$. Furthermore, it is clear from the maximality of N that the dimension of T is exactly one.

Now let $K(N) = K(L) \cap N$. From Iwasawa [4] $K(N)$ is a maximal compact subgroup of N . L' is contained in N , but L' is not closed in N . Therefore, L' does not contain $K(N)$ by Goto [2]. Let M be a maximal analytic subgroup of N containing L' , but not $K(N)$. If $M \neq \bar{M}$ we can show as before that $N/\bar{M}$ has dimension zero by the maximality of M , that is, $N = \bar{M}$ and $L = \bar{M} \cdot T$. Hence, MT is our desired proper dense analytic subgroup of L . Therefore, we may suppose that $M = \bar{M}$.

We have $K(N) = K'(N) \cdot Z_0(K(N))$, where $K'(N) \cap Z_0(K(N))$ is finite. Let T_2 denote the identity component group of $Z_0(K(N)) \cap M$. Since $K'(N)$ is contained in M , but $K(N)$ is not contained in M , there is a toral group T_3 of dimension one so that $Z_0(K(N)) = T_2 \cdot T_3$, $T_2 \cap T_3 = \{e\}$, and MT_3 is a closed normal analytic subgroup of N with $M \cap T_3$ finite. Since MT_3 contains both L' and $K(N)$ (by the maximality of M), N/MT_3 is a vector group.

From Lemma 1 and the maximality of M we can see as before that $N = MT_3$. Hence $L = NT = MT_3T$. Since $K(N) = K(L) \cap N$ and since T

is contained in $Z_0(K(L))$, we see that $T_4 = T_3 \cdot T$ is a two dimensional toral group and $L = MT_4$, where $M \cap T_4$ is finite. Letting V be a dense vector subgroup of T_4 we see that MV is a proper dense analytic subgroup of L , and $L = MV = MT_4$, thus completing the second case, and, therefore, the sufficiency part of our proof.

(ii) To prove necessity let us suppose now that L contains a proper dense analytic subgroup G . If L' is closed in L , then the following theorem of Goto [3] completes our proof. (We have altered the original statement of Goto's theorem, but a careful inspection of his proof justifies this alteration.)

Let G be a dense analytic subgroup of an analytic group L and suppose that G contains a maximal normal analytic subgroup N which contains the commutator subgroup of G and is also closed in L . Then for each maximal compact subgroup K of L there exists a closed vector subgroup V of G , such that $G = NV$, $N \cap V = \{e\}$, and $L = N\bar{V}$, where $N \cap \bar{V}$ is finite and $\bar{V}$ is a toral subgroup of L which is central in K .

COROLLARY 1. *Let L be a solvable linear analytic group. Then L possesses a proper dense analytic subgroup if and only if it contains a two dimensional toral subgroup.*

PROOF. The proof is a consequence of the above theorem and the well-known fact that the commutator subgroup of a linear analytic group is closed and the commutator subgroup of a solvable linear analytic group is homeomorphic to Euclidean space.

COROLLARY 2. *Let L be a nilpotent linear analytic group. Then L possesses a proper dense analytic subgroup if and only if $Z(L)$ contains a two dimensional toral subgroup.*

PROOF. The proof follows from Corollary 1 and the fact that any compact subgroup of L will be contained in $Z(L)$, since the adjoint group of L is homeomorphic to Euclidean space.

3. Examples.

EXAMPLE 1. Let $L = SL(2, \mathbf{R}) \times T$, where T is a one dimensional toral group. Then $K(L) = SO(2) \times T$ is a maximal compact subgroup of L which is a two dimensional toral group. But $K(L') = SO(2)$ is a maximal compact subgroup of L' . Therefore, L does not contain a proper dense analytic subgroup. We see that we cannot remove "solvable" from the hypothesis of Corollary 1.

EXAMPLE 2. Let G denote the nilpotent analytic group of 3×3 matrices of the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}; \quad a, b, c \in \mathbf{R}.$$

$Z(G)$ consists of all matrices for which $a = b = 0$. Let H denote the discrete

subgroup of $Z(G)$ consisting of all matrices for which c is an integer. Let $L = G/H \times G/H$. Then L is a nonlinear nilpotent analytic group in which $Z(L) = L' = K(L)$, where $K(L)$, the maximal compact subgroup of L , is a two dimensional toral group. From our theorem L does not contain a proper dense analytic subgroup. We see, therefore, that we cannot remove "linear" from either the hypothesis of Corollary 1 or the hypothesis of Corollary 2.

EXAMPLE 3. Let L denote the solvable analytic group of matrices of the form

$$\begin{pmatrix} e^{2\pi ir} & 0 & \alpha \\ 0 & e^{2\pi is} & \beta \\ 0 & 0 & 1 \end{pmatrix}; \quad r, s \in \mathbf{R}, \quad \alpha, \beta \in \mathbf{C}.$$

Let G denote the dense analytic subgroup of L consisting of those matrices for which $s = \gamma r$, where γ is a fixed irrational number. We see that $Z(L) = \{e\}$. Therefore, we cannot remove "nilpotent" from the hypothesis of Corollary 2.

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