PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 72, Number 3, December 1978

ON THE COVERING DIMENSION OF SUBSPACES OF PRODUCT OF SORGENFREY LINES[1]

ALI A. FORA

ABSTRACT. Let S denote the Sorgenfrey line. Then the following results are proved in this paper:

(i) If X is a nonempty subspace of $S^{\aleph_0}$, then $\dim X = 0$.

(ii) For any nonempty separable space $X \subset S^{\aleph_0}$, $\dim X^m = 0$ for any cardinal m.

1. Introduction. The question of whether $\dim(X \times Y) \leqslant \dim X + \dim Y$ for topological spaces X and Y has long been considered (see e.g., [**G**, pp. 263 and 277]). By $\dim X$, or the covering dimension of X, we mean the least integer, n, such that each finite cozero cover of X has a finite cozero refinement of order n. (A cover is of *order n* if and only if each point of the space is contained in at most $n + 1$ elements of the cover. All spaces considered are completely regular.)

Researchers have long worked on the above problem, and only recently Wage [**W**] and Przymusiński [**P**] constructed a Lindelöf space X such that $\dim X = 0$ and X^2 is normal but has $\dim X^2 > 0$.

The aim of this paper is to prove that no product of subspaces of Sorgenfrey lines can serve as a counterexample to the product conjecture. Another aim is to give a full answer to one of the questions raised by Mrowka [**Mr2**] in the conference of 1972 which says: "We still do not know if subspaces of S^n ($n = 2, 3, \ldots, \aleph_0$) are strongly 0-dimensional."

The familiar Sorgenfrey space S is defined to be the space of real numbers with the class of all half-open intervals $[a, b)$, $a < b$, as a base. It is a well-known fact that S is Lindelöf, first countable, N-compact and also has $\dim S = 0$.

A Tychonoff space X is called strongly zero-dimensional provided that $\dim X = 0$.

The following theorem (see e.g., [**G**]) characterizes the class of all strongly zero-dimensional spaces.

1.1 THEOREM. *For a Tychonoff space X, the following conditions are equivalent*:

Received by the editors August 9, 1976 and, in revised form, October 19, 1977.

AMS (MOS) subject classifications (1970). Primary 54F45; Secondary 54F50.

Key words and phrases. Completely regular, cozero cover, cozero set, N-compact, order of a cover, Tychonoff.

[1]The results in this paper are contained in the author's doctoral dissertation, written at SUNY at Buffalo under the direction of Stanislaw Mrowka in February 1976.

© American Mathematical Society 1978

(a) *X is strongly zero-dimensional.*
(b) *βX is strongly zero-dimensional.*
(c) *Every cozero-set of X is a countable union of clopen sets of X.*

It can be easily seen now that a Lindelöf space which has a base consisting of clopen sets must be strongly zero-dimensional. Since S^2 fails to be Lindelöf, there is no easy way to determine dim S^2. The fact that dim $S^n = 0$ for all n was proved only in 1972 **[Mr1]**, **[Te1]**. Prior to that, several researchers have proved that dim $S^2 = 0$ [Nyikos, Fund. Math. **79** (1973), 131–139], but their arguments could not be generalized, even to S^3. An interesting parallel is that Terasawa (private communication) has shown that S^2 is hereditarily strongly zero-dimensional; his proof cannot be generalized even to S^3.

2. The covering dimension of subspaces of product of Sorgenfrey lines. It is the time now to discuss the main result of this paper.

2.1 PROPOSITION. *Let Y be any strongly zero-dimensional metrizable space and $X \neq \varnothing$ be a subspace of $S^n \times Y$ (where n is a fixed integer). Then X is strongly zero-dimensional.*

PROOF. Let X be a nonempty subspace of $S^n \times Y$. By Theorem 1.1, it is sufficient to prove that each cozero-set of X is the union of countably many clopen sets in X.

The proof will be carried out in several steps (I–II).

I. For any $z = (z_1, \ldots, z_n) \in S^n$, define

$$V_i(z) = [z_1, z_1 + 1/i) \times \cdots \times [z_n, z_n + 1/i).$$

For each integer $i \geq 1$, choose the sequence $\{u_{ik}\}$ in S^n such that $S^n = \bigcup_{k=1}^{\infty} V_i(u_{ik})$. For simplicity, we let $V(i, k) = V_i(u_{ik})$, and, for $z = (z_1, \ldots, z_r)$, $A \subset S^n$, $B \subset Y$, we let $A \cap_\times B = (A \times B) \cap X$, $z(k) = z_k$ $(k \leq r)$.

Since dim $Y = 0$, therefore Y has a basis $\mathfrak{B} = \bigcup_{i=1}^{\infty} \mathfrak{B}_i$ consisting of clopen sets, where each $\mathfrak{B}_i$ is a locally finite family (see **[E**, p. 291]).

II. Let U be any cozero set of X determined by a continuous function f: $X \to [0, 1]$ in such a way that $U = f^{-1}(0, 1]$. For natural numbers i, k, l, and for each $G \in \mathfrak{B}_i$, define

$$W_G(i, k, l) = \{x = (z, y) \in X \mid f(V_i(z) \cap_\times G(y)) \subset (1/l, 1], \text{ where } z = (z_1, \ldots, z_n) \in S^n, y \in Y \text{ and } G(y) = G\} \cap V(i, k) \cap_\times G; \quad (1)$$

$$O_G(i, k, l) = \bigcup \{V_i(z) \cap_\times G(y) \mid (z, y) \in W_G(i, k, l)\} \cap V(i, k) \cap_\times G. \quad (2)$$

(3) If $(z_1, y_1) \in O_G(i, k, l)$ and $(z_2, y_2) \in V(i, k) \cap_\times G$ such that $z_2(j) \geq$

$z_1(j)$ $(j = 1, \dots, n)$, then $(z_2, y_2) \in O_G(i, k, l)$.

Define

$$\hat{O}_G(i, k, l) = \mathrm{cl}_X\, O_G(i, k, l) \setminus O_G(i, k, l). \tag{4}$$

One can notice that $\hat{O}_G(i, k, l) \subset V(i, k) \cap_\times G$, and that the closure of $O_G(i, k, l)$ is the same as its Euclidean closure in $V(i, k) \cap_\times G$.

One can also notice that

(5) if $(z_1, y_1) \in \hat{O}_G(i, k, l)$ and $(z_2, y_2) \in V(i, k) \cap_\times G$ such that $z_2(j) \geqslant z_1(j)$ $(j = 1, \dots, n)$, then $(z_2, y_2) \in \mathrm{cl}_X\, O_G(i, k, l)$.

Define:

(6) $T_G(i, k, l) = \bigcup \{N(x)\colon x \in \hat{O}_G(i, k, l)$ and $f(N(x)) \subset (1/(l + 1), 1]$, where $N(x)$ is a basic neighborhood at the point $x\} \cap V(i, k) \cap_\times G$;

$$F_G(i, k, l) = \mathrm{cl}_X\, O_G(i, k, l) \cup T_G(i, k, l) \subset V(i, k) \cap_\times G. \tag{7}$$

Then $F_G(i, k, l)$ is a clopen subset of X (see Observation 1 below).

Define

$$F(i, k, l) = \bigcup \{F_G(i, k, l) | G \in \mathfrak{B}_i\}.$$

Then $F(i, k, l)$ is clopen in X because $\mathfrak{B}_i$ is locally finite.

We can easily prove that $U = \bigcup \{F(i, k, l) | i, k, l\}$ (see Observation 2).

Observation 1. *$F_G(i, k, l)$ is a clopen subset of X.*

It suffices to prove that $F_G(i, k, l)$ is closed since it is clearly open. To show that $F_G(i, k, l)$ is closed, it suffices to prove that $\mathrm{cl}_X\, T_G(i, k, l) \setminus T_G(i, k, l) \subset F_G(i, k, l)$. Let $(z, y) \in \mathrm{cl}_X\, T_G \setminus T_G$. If $(z, y) \notin \mathrm{cl}_X\, O_G(i, k, l)$, there exists a basic neighborhood N_1 at (z, y) such that $(z, y) \in N_1 \subset V(i, k) \cap_\times G$ and $N_1 \cap O_G(i, k, l) = \varnothing$. Since $(z, y) \in \mathrm{cl}_X\, T_G \setminus T_G$, there exists a point $(z_1, y_1) \in N_1 \cap T_G$. From (6), we can find $(z_2, y_2) \in \hat{O}_G(i, k, l)$ such that $(z_1, y_1) \in N((z_2, y_2))$. Using (5), we get $(z_1, y_1) \in \mathrm{cl}_X\, O_G(i, k, l)$. Therefore $N_1 \cap O_G(i, k, l) \neq \varnothing$, which is a contradiction. Therefore $(z, y) \in \mathrm{cl}_X\, O_G(i, k, l)$ and consequently $(z, y) \in F_G(i, k, l)$.

Observation 2. $U = \bigcup \{F(i, k, l) | i, k, l\}$.

Let $(z, y) \in U \cap S^n \times Y$. By continuity of f, there exist l_0, $i_0 \geqslant 1$ and $G \in \mathfrak{B}_{i_0}$ such that $f(V_{i_0}(z) \cap_\times G(y)) \subset (1/l_0, 1]$, where $G(y) = G$. Let $k_0 \geqslant 1$ be such that $z \in V(i_0, k_0)$. Then $(z, y) \in F_G(i_0, k_0, l_0)$ which completes the proposition. We can proceed now to our main theorem.

2.2 THEOREM. *Let Y be any strongly zero-dimensional metrizable space and $X \neq \varnothing$ be a subspace of $S^{\aleph_0} \times Y$. Then X is strongly zero-dimensional.*

PROOF. Let U be any cozero set of X determined by a continuous function $f\colon X \to [0, 1]$ in such a way that $U = f^{-1}(0, 1]$. Then $U = \bigcup_{j=1}^{\infty} U_j$, where $U_j = \{x = (x_1, x_2, \dots, y) \in U\colon\ f(N_1 \times \cdots \times N_j \times S^{\aleph_0} \times G(y) \cap X) \subset (0, 1]$, where $N_1 \times \cdots \times N_j \times S^{\aleph_0} \times G(y) \cap X$ is some basic neighborhood at the point $x\}$ for $j = 1, 2, \dots$. For each natural number $j \geqslant 1$, write U_j as a countable union of clopen sets (use the same construction as in

Proposition 2.1). It is clear that $U = \bigcup_{j=1}^{\infty} U_j$ can be written as a countable union of clopen sets which completes the proof of the theorem.

We shall list various corollaries to the above theorem.

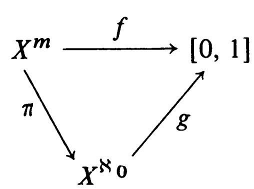
2.3 Corollary. *The space $S^m \times Y$ is hereditarily strongly zero-dimensional for every strongly zero-dimensional metrizable Y and all $1 \leqslant m \leqslant \aleph_0$.*

2.4 Corollary. *If $X \neq \varnothing$ is a separable subspace of $S^{\aleph_0} \times Y$, then X^m is strongly zero-dimensional for all cardinals m.*

Proof. Let Λ: $N \to N \times N$ be a fixed bijection. Define the map $\sim$: $(S^{\aleph_0} \times Y)^{\aleph_0} \to S^{\aleph_0} \times Y^{\aleph_0}$ by the rule $\sim (x_1, x_2, \ldots) = (x_{\Lambda 1}, x_{\Lambda 2}, \ldots, y_1, y_2, \ldots)$, where $x_i = (x_{1,i}, x_{2,i}, \ldots, y_i) \in S^{\aleph_0} \times Y$ for $i = 1, 2, \ldots$. It is clear that $\sim$ is a homeomorphism from $(S^{\aleph_0} \times Y)^{\aleph_0}$ onto $S^{\aleph_0} \times Y^{\aleph_0}$.

Let $\sim (X^{\aleph_0}) = \tilde{X}$. Then $\tilde{X} \subset S^{\aleph_0} \times Y^{\aleph_0}$ is strongly zero-dimensional and hence $X^{\aleph_0}$ is also strongly zero-dimensional.

Now, let m be any cardinal $\geqslant \aleph_0$, and let U be any cozero set of X^m which is determined by a continuous map f: $X^m \to [0, 1]$ in such a way that $U = f^{-1}(0, 1]$. By the Gleason Theorem (see [I]), we get the existence of continuous maps g and π such that the following diagram commutes.



It is clear that $U = \pi^{-1} (g^{-1}(0, 1])$ is a countable union of clopen sets in X^m.

2.5 Corollary. *If $\varnothing \neq X \subset S$, then X^m is strongly zero-dimensional for any cardinal m.*

The proof follows immediately from Corollary 2.4 and the fact that S is a hereditarily separable space.

3. Significance of the main results. As we explained, our results were prompted by the product conjecture "The product of any two strongly zero-dimensional spaces is still strongly zero-dimensional." However, these results are also relevant for other problems, e.g. for the problem of hereditary strong zero-dimensionality of various product spaces, strong zero-dimensionality of N-compact spaces and also for the following:

The Union Problem (U). If X_1 and X_2 are disjoint strongly zero-dimensional subspaces of X, with X_1 closed in X, and $X = X_1 \cup X_2$, then is X also strongly zero-dimensional?

We wish to discuss our results in view of the above problems. It has been recently demonstrated that N-compact spaces need not be strongly zero-dimensional. The first example was given in 1972 by Mrowka **[Mr2]**, and another one, still unpublished, by E. Pol and R. Pol. Both of these examples are quite complex; it is therefore reasonable to inquire whether some well-known space can serve as an example.

As we mentioned before, the product conjecture has been solved negatively by Wage **[W]** and Przymusiński **[P]**. The Union Problem also has been solved negatively by Terasawa **[Te2]**.

It is easy to see that $S^{\aleph_0}$ is hereditarily N-compact. Consequently, for some time, it was conjectured that subspaces of $S^{\aleph_0}$ could provide an example of N-compact nonstrongly zero-dimensional space. Our results eliminate this possibility; more exactly, they eliminate the following spaces from these considerations:

(a) subspaces of $S^{\aleph_0}$,
(b) products of subspaces of S.

Observe also that, for large n, S^n is not even closed hereditary strongly zero-dimensional. Indeed, if we take the nonstrongly zero-dimensional N-compact space μ (see **[Mr2]**), then, for large n, we have $\mu \subset_{\text{cl}} N^n \subset_{\text{cl}} S^n$.

Moreover, our results may be considered as generalizations to those found in **[Mr1]** and **[Te1]**.

To conclude this section we will comment on the connection of our results with the so-called intermediate topology (described in **[Ta]**). These matters are related to both the Product and the Union Problems. In this matter, one considers a space X with a distinguished subset M such that M and $X \setminus M$ are both metrizable strongly zero-dimensional spaces; the theorem in **[Ta]** asserts that, under a certain additional assumption, X is strongly zero-dimensional.

On the other hand, by our results, $M \times S^{\aleph_0}$ and $(X \setminus M) \times S^{\aleph_0}$ are both strongly zero-dimensional. Thus, if $X \times S^{\aleph_0}$ (with an X satisfying the assumption of the required theorem in **[Ta]**) would fail to be strongly zero-dimensional, this would provide a counterexample to the Product Problem as well as to the Union Problem.

REFERENCES

[E] R. Engelking, *Outline of general topology*, North-Holland, Amsterdam, 1968.

[G] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, New York, 1960.

[I] J. R. Isbell, *Uniform spaces*, Math. Surveys, American Mathematical Society, Providence, R.I., 1964.

[Mr1] S. Mrowka, *Recent results on E-compact spaces and structures of continuous functions* (Proc. Univ. of Oklahoma Topology Conf., 1972), Univ. of Oklahoma, Norman, 1972, pp. 168–221.

[Mr2] ______, *Recent results on E-compact spaces*, TOPO–General Topology and its

Applications, Lecture Notes in Math., vol. 378, Springer, Berlin and New York, 1974, 298–301.

[P] T. C. Przymusiński, *On the dimension of product spaces and an example of M. Wage* (to appear).

[Ta] H. P. Tan, Doctoral Dissertation, Buffalo University, 1973.

[Te1] J. Terasawa, *On the zero-dimensionality of some nonnormal product spaces*, Sci. Rep. Tokyo Kyoiku Daigaku **11** (1972), 167–174.

[Te2] ______, *NUR and their dimensions* (to appear).

[W] M. Wage, *The dimension of product spaces* (to appear).

Department of Mathematics, Yarmouk University, Irbid, Jordan