A TOPOLOGICAL NOTE ON THE ADAMS SPECTRAL SEQUENCE BASED ON MORAVA'S K -THEORY

NOBUAKI YAGITA

ABSTRACT. In this note, we compute the Hopf algebra $K(n)_*(K(n))$ and consider the Adams spectral sequence of $K(n)_*(-)$ theory.

1. Introduction. Miller and Ravenel [3] showed that

$$\text{Ext}_{\text{BP}_*\text{BP}}(\text{BP}_* [v_n^{-1}] \text{BP}_* / I_n) \cong \text{Ext}_{K(n)_*K(n)}(K(n)_*, K(n)_*).$$

Miller, Morava, Ravenel and Wilson [3], [4], [5] make use of the right side group to compute stable homotopy groups of spheres. However, the group is *not straightway* induced by the Adams spectral sequence based on Morava's extraordinary $K(n)$ -theory [5]; in fact $K(n)_*K(n)$ is purely algebraically defined and is not isomorphic to $\pi_*(K(n) \wedge K(n)) = K(n)_*(K(n))$.

In this paper, however, we show that the group is isomorphic to E_2 -term of the spectral sequence, i.e., if $V(n-1)$ exists,

$$\text{Ext}_{K(n)_*K(n)}(K(n)_*, K(n)_*) \cong \text{Ext}_{K(n)_*(K(n))}(K(n)_*(S^0), K(n)_*(V(n-1))).$$

2. $[v_m^{-1}] \text{BP}(S)_*(\text{BP}(S))$. Let $\text{BP}_*(-)$ be the Brown-Peterson homology theory at a fixed odd prime p , with the coefficient $\text{BP}_* \cong \mathbb{Z}_{(p)}[v_1, \dots]$. By using the Sullivan's bordism theory with singularities [9], we can construct the homology theory $\text{BP}(S)_*(-)$ with the coefficient $\text{BP}_*/(S)$ where $S = (P_1, \dots)$ is a regular sequence of $P_i \in \text{BP}_*$. Since $\text{BP}(S)$ is a ring spectrum [7], $\text{BP}(S)_*(\text{BP}(S))$ is a BP_* -algebra. Note that $\text{BP}_*(\text{BP}) \cong \text{BP}_*[t'_1, \dots]$ and $H_*(K\mathbb{Z}_p; \mathbb{Z}_p) \cong \mathbb{Z}_p[t'_1, \dots] \otimes \Lambda[\tau'_0, \dots]$ (see [1], [10]). Let $i_1: \text{BP}_*(\text{BP}) \rightarrow \text{BP}(S)_*(\text{BP}) \rightarrow \text{BP}(S)_*(\text{BP}(S))$ and $i_2: \text{BP}(S)_*(\text{BP}(S)) \rightarrow H_*(\text{BP}(S); \mathbb{Z}_p) \rightarrow H_*(K\mathbb{Z}_p; \mathbb{Z}_p)$ be maps induced by natural inclusion maps $i_1: \text{BP} \rightarrow \text{BP}(S)$ and $i_2: \text{BP}(S) \rightarrow K\mathbb{Z}_p$.

THEOREM 1. Let $S = (v_{j_0} = p, v_{j_1}, \dots)$ and $j_k < j_{k+1}$. If $j_k = k$ for $0 \leq k \leq m-1$ and $j_m \neq m$, then we have a BP_* -algebra isomorphism

$$\begin{aligned} [v_m^{-1}] \text{BP}(S)_*(\text{BP}(S)) \\ \cong [v_m^{-1}] \text{BP}_*[t_1, \dots] \otimes \Lambda[\tau_0, \dots, \tau_{m-1}] / (S, \eta_R(S)) \end{aligned}$$

where t_i, τ_i are elements in $\text{BP}(S)_*(\text{BP}(S))$ so that $i_1(t'_i) = t_i$, $i_2(\tau'_i) = \tau'_i$ and $(S, \eta_R(S))$ is the ideal generated by $(v_{j_0}, \dots, \eta_R(v_{j_0}), \dots)$.

Received by the editors March 1, 1977 and, in revised form, September 6, 1977.

AMS (MOS) subject classifications (1970). Primary 55B20, 55H15; Secondary 55G25.

© American Mathematical Society 1978

PROOF. Let $S_n = (v_{j_0}, \dots, v_{j_n})$. Since $\mathrm{BP}(S_n)$ is constructed by the cofiber map

$$\mathrm{BP}(S_{n-1}) \xrightarrow{v_{j_n}} \mathrm{BP}(S_{n-1}) \rightarrow \mathrm{BP}(S_n),$$

there is the BP_*/S -module exact sequence

$$\begin{array}{ccc} \mathrm{BP}(S)_*(\mathrm{BP}(S_{n-1})) & \xrightarrow{v_{j_n} = 1 \wedge v_{j_n}} & \mathrm{BP}(S)_*(\mathrm{BP}(S_{n-1})). \\ \delta_* \swarrow & & \searrow i_* \\ & \mathrm{BP}(S)_*(\mathrm{BP}(S_n)) & \end{array}$$

Here, we note that by using the Sullivan (Bockstein) exact sequence [7], [9], we can easily see that $\mathrm{BP}(S)_*(\mathrm{BP}) \cong \mathrm{BP}_*/(S) \otimes_{\mathrm{BP}_*} \mathrm{BP}_*(\mathrm{BP})$, since BP is torsion free and S is regular.

Assume $n < m$. Then, in $\mathrm{BP}_*(\mathrm{BP})$, $1 \wedge v_{j_n} = \eta_R(v_{j_n}) = 0 \pmod{(p, \dots, v_{m-1}) \subset (S)}$. By induction and compatibility of the product, we can prove that $v_{j_n} = 0$ in $\mathrm{BP}(S)_*(\mathrm{BP}(S_{n-1}))$ (see the proof of Theorem 3.4 in [9]). And let $\tau_n \in \delta_*^{-1}(1)$; then we can prove

$$\mathrm{BP}(S)_*(\mathrm{BP}(S_n)) \cong \mathrm{BP}_*/(S) \otimes_{\mathrm{BP}_*} \mathrm{BP}_*(\mathrm{BP}) \otimes \Lambda[\tau_0, \dots, \tau_n],$$

and moreover, we can choose τ_i so that $i_3(\tau_i) = \tau'_i$ where $i_3: \mathrm{BP}(S)_*(\mathrm{BP}(S_n)) \rightarrow H_*(KZ_p; Z_p)$ is the map induced by the natural inclusion map $i_3: \mathrm{BP}(S_n) = \mathrm{BP}(p, \dots, v_{j_n}) \rightarrow KZ_p$ (see [9]).

Assume $n \geq m$. By [6] in $\mathrm{BP}_*(\mathrm{BP})$ we have

$$\begin{aligned} 1 \wedge v_{j_n} &= \eta_R(v_{j_n}) = v_m t_j'^{p^m} - v_m^{p^j} t_j' \\ &\pmod{(p, \dots, \hat{v}_m, \dots, \eta_R(v_{m+1}), \dots, \eta_R(v_{j_n-1}))} \\ &\subset (p, \dots, \hat{v}_m, \dots, t'_1, \dots, t'_{j-1}). \end{aligned}$$

where $j = j_n - m$. Therefore $\eta_R(v_{j_n})$ is a non zero divisor of $[v_m^{-1}] \mathrm{BP}_*(\mathrm{BP})/(S, \eta_R(S_{n-1}))$. By induction and also the compatibility of the product, we can prove that v_{j_n} is injective in $[v_m^{-1}] \mathrm{BP}(S)_*(\mathrm{BP}(S_{n-1}))$, therefore the proof is completed.

REMARK 1. By the exact functor theorem for $\mathrm{BP}_*/(p, \dots, v_{m-1})$ [11] we have

$$[v_m^{-1}] \mathrm{BP}(S)_*(-) \cong [v_m^{-1}] \mathrm{BP}_*/S \otimes_{\mathrm{BP}_*} \mathrm{BP}(p, \dots, v_{m-1})_*(-).$$

Using this fact, we can also prove Theorem 1.

By arguments similar to the proof of Theorem 1, we can prove that

$$\begin{aligned} [v_m^{-1}] \mathrm{BP}(S)_*(\mathrm{BP}(S) \wedge \mathrm{BP}(S)) \\ \cong [v_m^{-1}] \mathrm{BP}(S)_*(\mathrm{BP}(S)) \otimes_{\mathrm{BP}_*} \mathrm{BP}(S)_*(\mathrm{BP}(S)). \end{aligned}$$

Hence $[v_m^{-1}]BP(S)_*(BP(S))$ is a Hopf algebra in the standard way [1].

THEOREM 2. Let ψ be the coaction map in $[v_m^{-1}]BP(S)_*(BP(S))$.

(1) $\sum^k \psi(t_k) = \sum^k t_i \otimes t_{k-i}^{p^i}$ where μ is the product of the formal group law of BP_* .

(2) $\psi(\tau_k) = \sum_{i \geq 0} \tau_i \otimes t_{k-i}^{p^i} + 1 \otimes \tau_k$.

PROOF. By the coaction of $BP_*(BP)$ [1], [6] and the compatibility of the product, (1) is immediate. Let $i_4: BP(S_{m-1})_*(BP(S_{m-1})) \rightarrow BP(S)_*(BP(S))$ be the natural inclusion map. Since $[v_m^{-1}]i_4$ is epic by Theorem 1, we may prove that (2) holds in $BP(S_{m-1})_*(BP(S_{m-1}))$. Let $i_5: BP(S_{m-1})_*(BP(S_{m-1})) \rightarrow H_*(KZ_p; Z_p)$ be the natural inclusion map. Then by Theorem 3 in [10], we have $i_5(\psi(\tau_k) - (2)) = 0$. But $\ker i_5 = \text{Ideal}(v_m, v_{m+1}, \dots)$. Since $\dim \tau_k = 2p^k - 1 < 2(p^m - 1)$, we have $\psi(\tau_k) = (2)$.

EXAMPLE 1. Denote $BP(p, \dots, v_{n-1})$ by $P(n)$ and $BP(p, \dots, \hat{v}_n, \dots)$ by $k(n)$ due to notations in [2]–[4].

$$P(n)_*(P(n)) \cong P(n)_* \otimes_{BP_*} BP_*(BP) \otimes \Lambda[\tau_0, \dots, \tau_{n-1}],$$

$$[v_n^{-1}]k(n)_*(k(n)) \cong [v_n^{-1}]k(n)_* \otimes_{BP_*} BP_*(BP)$$

$$\otimes \Lambda[\tau_0, \dots, \tau_{n-1}] / (v_n t_i^{p^n} - v_n^{p^i} t_i).$$

3. $\text{Ext}_{K(n)_*(K(n))}(K(n)_*(S^0), K(n)_*(V(n-1)))$. Let $K(n)_*(-) = [v_n^{-1}]k(n)_*(-)$ [2], [3], [4]. Then we have

$$\begin{aligned} K(n)_*(K(n)) &= [v_n^{-1}]k(n)_*(K(n)) \cong [v_n^{-1}]c(K(n)_*(k(n))) \\ &\cong [v_n^{-1}][\eta_R(v_n)^{-1}]c(k(n)_*(k(n))) \cong [v_n^{-1}]k(n)_*(k(n)) \end{aligned}$$

where c is the conjugation map.

On the other hand, in [3], [4], [6], $K(n)_*K(n)$ (without parentheses) is defined by

$$K(n)_*K(n) = K(n)_* \otimes_{BP_*} BP_*(BP) / (v_n t_i^{p^n} - v_n^{p^i} t_i).$$

Hence we have

$$K(n)_*(K(n)) \cong K(n)_*K(n) \otimes \Lambda[\tau_0, \dots, \tau_{n-1}].$$

LEMMA 1. Let L, R be $K(n)_*K(n)$ -comodules; then we have

$$\text{Ext}_{K(n)_*(K(n))}(L, R) \cong \text{Ext}_{K(n)_*(K(n))}(L, R \otimes \Lambda[\tau_0, \dots, \tau_{n-1}]).$$

PROOF. Let $0 \rightarrow R \rightarrow R_1 \rightarrow \dots$ be the $K(n)_*K(n)$ -cobar resolution. Then each R_i is an extended $K(n)_*K(n)$ -comodule. Then

$$0 \rightarrow R \otimes \Lambda[\tau_0, \dots, \tau_{n-1}] \rightarrow R_1 \otimes \Lambda[\tau_0, \dots, \tau_{n-1}] \rightarrow \dots$$

is a resolution so that each $R_i \otimes \Lambda[\tau_0, \dots, \tau_{n-1}]$ is an extended $K(n)_*(K(n))$ -comodule. Hence we need only prove that

$$\mathrm{Hom}_{K(n)_*K(n)}(L, R_i) \cong \mathrm{Hom}_{K(n)_*(K(n))}(L, R_i \otimes \Lambda[\tau_0, \dots, \tau_{n-1}]), \quad (1)$$

where Hom is the homomorphisms of comodules.

Let $f \in$ (the right side of (1)) and $a \in L$. Let $f(a) = \sum b_\alpha \tau^\alpha$ where $b_\alpha \in R_i$ and $\tau^\alpha = \tau_0^{\alpha_0} \tau_1^{\alpha_1} \dots$. Then $\psi f(a) = \sum \psi(b_\alpha) \psi(\tau^\alpha)$. Here $\psi(b_\alpha) \in R_i \otimes_{K(n)_*} K(n)_* K(n)$ and $\psi(\tau^\alpha) = 1 \otimes \tau^\alpha + \sum_{|\alpha'| < |\alpha|} \tau^{\beta} \otimes \tau^{\alpha'}$. But $\psi(f(a)) = f(\psi(a)) \in R_i \otimes_{K(n)_*} K(n)_* K(n)$. Hence if $|\alpha| > 0$ then $\psi(b_\alpha) = 0$, so $b_\alpha = 0$. Therefore we can consider $f \in$ (the left side of (1)).

Now consider the Adams spectral sequence (see §15 in [1]). Since $K(n)_*(K(n))$ satisfies assumptions (i), (ii) on p. 317 in [1], there exists a spectral sequence whose E_2 -term is

$$\mathrm{Ext}_{K(n)_*(K(n))}(K(n)_*(X), K(n)_*(Y))$$

for arbitrary spectra X, Y . However, the assumption (iii) is not satisfied; the spectral sequence doesn't necessarily converge to $\pi_*[X, Y]_{(p)}$.

Here, recall that $V(n-1)$ is the spectrum such that $\mathrm{BP}_*(V(n-1)) \cong \mathrm{BP}_*/(p, \dots, v_{n-1})$ [8]. Miller-Ravenel [3] showed

$$\mathrm{Ext}_{K(n)_*K(n)}(K(n)_*, K(n)_*) \cong \mathrm{Ext}_{\mathrm{BP}_*(\mathrm{BP})}(\mathrm{BP}_*, [\mathrm{BP}_*^{-1}] \mathrm{BP}_*/(p, \dots, v_{n-1})).$$

We have the following theorem.

THEOREM 3. *If $V(n-1)$ exists,*

$$\begin{aligned} \mathrm{Ext}_{K(n)_*K(n)}(K(n)_*, K(n)_*) \\ \cong \mathrm{Ext}_{K(n)_*(K(n))}(K(n)_*(S^0), K(n)_*(V(n-1))). \end{aligned}$$

PROOF. By Lemma 1, we need only prove that there is a $K(n)_*(K(n))$ -comodule isomorphism

$$K(n)_*(V(n-1)) \cong K(n)_* \otimes \Lambda[\tau_0, \dots, \tau_{n-1}].$$

If $V(n-1)$ exists, then for $0 \leq i \leq n-1$, there exists $V(i)$ and it is constructed by the following cofiber

$$V(i-1) \xrightarrow{v_i} V(i-1) \rightarrow V(i).$$

By arguments similar to the proof of Theorem 1, the proof is easy.

REMARK 2. In $k(n)_*(-)$ theory, assumptions (i), (iii) are satisfied, but (ii) is not. Hence, for connected spectra X, Y , there exists the spectral sequence E_i which converges to $\pi_*[X, Y]_{(p)}$. However, the E_2 -term seems to be complicated.

REFERENCES

1. J. F. Adams, *Stable homotopy and generalised homology*, Univ. Chicago Press, Chicago, Ill., 1974.
2. D. C. Johnson and W. S. Wilson, *BP operations and Morava's extraordinary K-theories*, Math. Z. **144** (1975), 55-75.

3. H. R. Miller and D. C. Ravenel, *Morava stabilizer algebras and the localization of Novikov's E_2 -term*, Duke Math. J. **44** (1977), 433–447.
4. H. R. Miller, D. C. Ravenel and W. S. Wilson, *Novikov's Ext^2 and the nontriviality of the gamma family*, Bull. Amer. Math. Soc. **81** (1975), 1073–1075.
5. J. Morava, *Structure theorems for cobordism comodules* (to appear).
6. D. C. Ravenel, *The structure of Morava stabilizer algebras*, Invent. Math. **37** (1976), 109–120.
7. N. Shimada and N. Yagita, *Multiplications in the complex bordism theory with singularities*, Publ. RIMS, Kyoto Univ. **12** (1976), 259–293.
8. L. Smith, *On realizing complex cobordism modules*, Amer. J. Math. **92** (1970), 793–856.
9. N. Yagita, *On the algebraic structure of cobordism operations with singularities*, J. London Math. Soc. **16** (1977), 131–141.
10. J. Milnor, *The Steenrod algebra and its dual*, Ann. of Math. (2) **67** (1958), 250–271.
11. N. Yagita, *The exact functor theorem for BP_*/I_n -theory*, Proc. Japan Acad. **52** (1976), 1–3.

DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY, OH-OKAYAMA MEGUROKU,
TOKYO, JAPAN