ABNORMALITIES IN NOETHERIAN RINGS

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ABSTRACT. If $P \subseteq Q$ are prime ideals in some ring R and if $\text{rank } Q = \text{rank}(Q/P) + \text{rank } P + k$, then $P \subset Q$ is said to be k -abnormal and k is called the degree of abnormality. The paper consists of two examples. The first example is a Noetherian integral domain in which the set of degrees of abnormality is unbounded. Let P be a prime ideal of R and set $W = \{Q \mid Q \text{ is a prime ideal and } P \subset Q \text{ is abnormal}\}$. The second example is a local domain such that $\{k \mid P \subset Q \text{ is } k\text{-abnormal for some } Q \in W\} \neq \{k \mid P \subset Q \text{ is } k\text{-abnormal for some } Q \text{ minimal in } W\}$.

Throughout the paper all rings considered are assumed to be commutative with unity. Let $P \subset Q$ be prime ideals in a Noetherian ring R . In studying the various chain conditions and related questions on R , it is frequently of interest to know whether it is the case that $\text{rank } Q = \text{rank}(Q/P) + \text{rank } P$. (Cf. Corollary 2.4, Corollary 2.13, and Remark 2.33(i) of [5].) It is well known that this equality need not hold in a Noetherian ring [4, Example 2, p. 203] and it thus becomes natural to consider abnormalities in a Noetherian ring. If $\text{rank } Q = \text{rank}(Q/P) + \text{rank } P + k$, then $P \subset Q$ is said to be k -abnormal and k is called the *degree of abnormality*. This notion is introduced and studied by E. G. Houston and S. McAdam in [2]. In [2, Theorem 1.6], Houston and McAdam prove for a given prime ideal P that $\{k \mid \text{there is a prime ideal } Q \text{ with } P \subset Q \text{ } k\text{-abnormal}\}$ is a finite set. They then ask whether the set of all degrees of abnormalities that can occur in a Noetherian ring is finite or, equivalently, whether there is a Noetherian ring in which the set of degrees of abnormalities is unbounded [2, p. 67]. In Example 1 we construct such a ring. For a given prime ideal P , Houston and McAdam also show in [2, Corollary 1.9] that the set $\{Q \mid Q \text{ is a prime ideal such that } P \subset Q \text{ is abnormal}\}$ has only finitely many minimal elements. They ask [2, pp. 67–68] whether, in a Noetherian ring, every degree of abnormality for any $P \subset Q$ occurs for some choice of Q among these minimal elements. Example 2 illustrates that this need not be the case even for local domains.

The examples presented in this paper rely heavily on Nagata's Example 2 [4, pp. 203–205]. The following proposition is essential to the execution of the construction.

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PROPOSITION. *Let $(T, M_1, \dots, M_n)$ be a semilocal domain containing a field K such that T/M_i is finite over K for all i . Then $R = K + M$, where $M = \bigcap_{i=1}^n M_i$, is a local domain with maximal ideal M . The contraction map from the ideals of T to those of R induces a homeomorphism from $\text{spec}(T) \setminus \{M_1, \dots, M_n\}$ onto $\text{spec } R \setminus \{M\}$. Furthermore, T is integral over R .*

This proposition is essentially the same as (E2.1) from Nagata [4, p. 204] except for our statement relating the prime spectrums of R and S which is an easy consequence.

We state here the essential steps in the construction of Nagata's Example 2. Let K be a field, let x be an indeterminate over K , and for $1 \leq i \leq r$ let $z_i = \sum_{j=0}^{\infty} a_{ij}x^j$ be elements in $K[[x]]$ that are algebraically independent over $K(x)$. Set $(z_i)_0 = xz_i$ for each i , and for $j > 0$ set

$$(z_i)_j = \left(z_i - \sum_{k < j} a_{ik}x^k \right) / x^{j-1}.$$

We denote by Z_i the set $\{(z_i)_j\}_{j=0}^{\infty}$. In the ring $R_1 = K[x, \{Z_i\}_{i=1}^r]$ the maximal ideals $M = (x)$ and $N = (x - 1, z_1, \dots, z_r)$ have rank 1 and rank $r + 1$, respectively. The basic notion of the construction is to now extend M and N to prime ideals M^e and N^e , respectively, in an appropriate ring T with maximal ideals M'' and N'' such that $M^e \subset M''$, $N^e \subset N''$, $\text{rank } M'' = 2$, and $\text{rank } N'' = r + 2$. Finally, the ideals M'' and N'' are "joined" as indicated in the above proposition to form a maximal ideal P in a local ring with saturated chains of prime ideals as follows:

$$\begin{array}{ccccc} & & Q & & \\ & \subset & & \subset & \\ (0) & & & & P \\ & \subset & & \subset & \\ & P_1 & \subset \cdots \subset & P_{r+1} & \end{array}$$

Specifically, let y be an indeterminate over R_1 and in $R_2 = R_1[y]$ let $M' = (x, y)$ and $N' = (x - 1, z_1, \dots, z_r, y)$. Then M' and N' are maximal ideals of rank 2 and rank $r + 2$, respectively. If S denotes the multiplicative system $R_2 \setminus (M' \cup N')$, then $((R_2)_S, M'_S, N'_S)$ is a semilocal domain with $(R_2)_S/M'_S \cong (R_2)_S/N'_S \cong K$. By the Proposition $R = K + (M' \cap N')_S$ is a local (Noetherian) ring in which $(M' \cap N')_S \supset x(R_2)_S \cap R \supset (0)$ is a saturated chain and $(M' \cap N')_S$ has rank $r + 2$. Thus, $x(R_2)_S \cap R \subset (M' \cap N')_S$ is abnormal of degree r .

It is evident that a ring in which the set of degrees of abnormality is unbounded has infinite Krull dimension. Nagata's Example 1 [4, p. 203] is the standard example of a Noetherian ring with infinite Krull dimension and the reader will observe that in our Example 1 we employ some of the techniques used by Nagata in his example.

EXAMPLE 1. *A Noetherian ring in which the set of degrees of abnormality is unbounded.*

Let K denote a field and let $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ be indeterminates over K .

For each i , let $z_{i1}, \dots, z_{ii}$ be elements in $K[[x_i]]$ that are algebraically independent over $K(\{x_i\}_{i=1}^\infty)$ and set $K_i = K(\{x_j, Z_{j1}, \dots, Z_{ji}, y_j\}_{j \neq i})$ where as in the preliminary remarks $Z_{jk} = \{(z_{jk})_t\}_{t=0}^\infty$. If we perform Nagata's construction using

$$D_i = K_i[x_i, Z_{i1}, \dots, Z_{ii}, y_i]$$

(D_i plays the role of the R_2 in the preliminary discussion), yielding

$$R_i = K_i + (M_i \cap N_i)_{S_i}$$

where $M_i = (x_i, y_i)D_i$, $N_i = (x_i - 1, z_{i1}, \dots, z_{ii}, y_i)D_i$ and $S_i = D_i \setminus (M_i \cup N_i)$, then, as observed above, R_i is a local ring and $[(x_i D_i)_{S_i} \cap R_i] \subset (M_i \cap N_i)_{S_i}$ has degree of abnormality i .

THEOREM. *The ring $R = \bigcap_{i=1}^\infty R_i$ is a Noetherian ring in which the set of degrees of abnormality is unbounded.*

PROOF. Our approach is to construct a ring T such that each R_i is a localization at some prime ideal of T . Thus, let T denote the ring $K[\{L_i\}_{i=1}^\infty]$ where $L_i = (M_i \cap N_i) \cap K[x_i, Z_{i1}, \dots, Z_{ii}, y_i]$. Since $T/L_i T \cong K[\{L_j\}_{j \neq i}]$, $L_i T$ is a prime ideal and so $W_i = T \setminus L_i T$ is a multiplicative system. We will show that $T_{W_i} = R_i = K_i + (M_i \cap N_i)_{S_i}$. Let $a/b \in K_i$ where $a, b \in K[\{x_j, Z_{j1}, \dots, Z_{ji}, y_j\}_{j \neq i}] \setminus (0)$. For each j , if $f \in K[x_j, Z_{ji}, \dots, Z_{jj}, y_j] \setminus (0)$, then $y_j f \in L_j \setminus (0) \subseteq W_i$ for $i \neq j$. Thus, if $k_1, \dots, k_s$ is a complete list of all the subscripts appearing in a or b , then

$$a/b = \left(a \prod_{t=1}^s y_{k_t} \right) / \left(b \prod_{t=1}^s y_{k_t} \right) \in T_{W_i}.$$

Therefore $K_i \subseteq T_{W_i}$. Since M_i and N_i are comaximal in D_i , $M_i \cap N_i = M_i N_i$. Since each of M_i and N_i is generated by elements of $K[x_i, Z_{i1}, \dots, Z_{ii}, y_i]$, it follows that $L_i D_i = M_i \cap N_i$. In fact, L_i generates $M_i \cap N_i$ as a vector space over K_i . Suppose that P is a prime ideal of D_i that contains no elements from the saturation of W_i in D_i , and that $P \not\subseteq M_i \cup N_i$. Since M_i and N_i are the only maximal ideals of D_i which contain $M_i \cap N_i$ and neither contains P , it follows that P and $M_i \cap N_i$ are comaximal. If $p \in P$ and $m \in M_i \cap N_i$ are such that $p + m = 1$, then $p = 1 - m \in T_{W_i} \setminus (L_i T)_{W_i}$ and so there exists $d \in W_i$ such that $dp \in T \setminus L_i T = W_i$. Since this contradicts our choice of P , we must have $P \subseteq M_i \cup N_i$ so $S_i = D_i \setminus (M_i \cup N_i)$ is the saturation of W_i in D_i .

We now have $K_i \subseteq T_{W_i}$ and $(M_i \cap N_i)_{S_i} = (L_i T)_{W_i}$, so it follows that $T \subseteq R_i \subseteq T_{W_i}$. But R_i is a local ring whose maximal ideal is centered on $L_i T$ in T . Therefore, $R_i = T_{W_i}$.

Now let $W = \bigcap_{i=1}^\infty W_i = T \setminus \bigcup_{i=1}^\infty L_i T$. We wish to show that $R = \bigcap_{i=1}^\infty R_i = T_W$. We begin by showing that if Q is a prime ideal of T such that $Q \cap W = \emptyset$ —that is, $Q \subseteq \bigcup_{i=1}^\infty L_i T$ —then $Q \subseteq L_i T$ for some i . It will then follow from [1, Lemma 5.4] that $R = T_W$. Thus, suppose that $Q \subseteq \bigcup_{i=1}^\infty L_i T$,

$Q \not\subseteq L_i T$ for all i , and let $a \in Q$, $a \neq 0$. Since a involves only finitely many variables it is contained in only finitely many of the ideals $L_j T$, say $a \in L_j T$ for $1 \leq j \leq k$ and $a \notin L_j T$ for $j > k$. If $Q \subseteq \bigcup_{j=1}^k (L_j T)$, then $Q \subseteq L_j T$ for some j , $1 \leq j \leq k$, contrary to our assumption. Therefore, there exists $b \in Q \setminus \bigcup_{j=1}^k L_j T$ and $(a, b)T \not\subseteq L_i T$ for all i . Let n and m be positive integers with $m > n$. No $L_i T$ contains both $a + b^n$ and $a + b^m$, for suppose $a + b^n, a + b^m \in L_i T$. In this case neither a nor b can be in $L_i T$ since otherwise they both would be, a contradiction of the choice of a and b . Consequently since

$$(a + b^n) - (a + b^m) = b^n - b^m = b^n(1 - b^{m-n}) \in L_i T$$

and $b \notin L_i T$ it follows that $1 - b^{m-n} \in L_i T$. Since $b \in Q \subseteq \bigcup_{j=1}^{\infty} L_j T$, b has zero constant term and $1 - b^{m-n}$ has constant term 1. But no such element can be contained in $\bigcup_{j=1}^{\infty} L_j T$'s. Hence $a + b^n$ and $a + b^m$ are in distinct $L_i T$'s. This shows that the set $\{a + b^n\}_{n=1}^{\infty} \subseteq Q \subseteq \bigcup_{j=1}^{\infty} L_j T$ cannot be contained in any finite union of the $L_j T$. But, to the contrary, since a and b each involve only a finite number of indeterminates, $\{a + b^n\}_{n=1}^{\infty}$ is contained in only finitely many $L_j T$'s. We now have the desired conclusion that $Q \subseteq L_i T$ for some i . Hence

$$T_w = \bigcap_{i=1}^{\infty} T_{L_i T} = \bigcap_{i=1}^{\infty} T_{w_i} = \bigcap_{i=1}^{\infty} R_i = R.$$

Since $R_i = T_{w_i} = (T_w)_{(L_i T)_w} = R_{(L_i T)_w}$, R is locally Noetherian and has an abnormality of degree i for each positive integer i . If $f \in R$, $f \neq 0$, then as observed above, f is contained in only a finite number of maximal ideals. By [4, E1.1, p. 203] R is a Noetherian ring and the proof is complete.

EXAMPLE 2. *A local domain with prime ideals $I_1 \subset I_2$ that are abnormal of degree 2 but such that no minimal element of $\{I | I \text{ is a prime ideal and } I_1 \subset I \text{ is abnormal}\}$ is 2-abnormal over I_1 .*

Let K be a field of characteristic 2 with $|K| > 2$ and let $x, w, z \in K[[x]]$ be algebraically independent over K . We write

$$w = \sum_{i=0}^{\infty} a_i x^i \quad \text{and} \quad z = \sum_{i=0}^{\infty} b_i [x(x+1)]^i$$

and define $w_0 = xw$, $w_i = (w_{i-1}/x) - a_{i-1}$ for $i > 0$. Similarly, we define

$$z_0 = [x(x+1)]z \quad \text{and} \quad z_i = [z_{i-1}/x(x+1)] - b_{i-1}$$

for $i > 0$. If $R_1 = K[x, \{w_i\}_{i=0}^{\infty}, \{z_i\}_{i=0}^{\infty}]$ then $M_1 = (x)$, $M_2 = (x+1, w)$, and $M_3 = (x+\beta, w, z)$, $\beta \neq 0, 1$, are maximal ideals of R_1 such that $R_1/M_i = K$ for $i = 1, 2, 3$. Note that $(R_1)_{(x)}$ is a discrete rank one valuation ring (DVR) (since $(R_1)_{(x)}$ is dominated by $K[[x]]$ —cf. [4, p. 204]).

Claim. $(R_1)_{M_2}$ is a two dimensional Noetherian ring.

For the same reason that $(R_1)_{(x)}$ is a DVR, $K[x, \{z_i\}_{i=0}^{\infty}]_{(x)}$ is a DVR. Consider the automorphism σ on $K(x, z)$ such that $\sigma(x) = x+1$ and $\sigma(z) = z$. Since $\sigma(x(x+1)) = x(x+1)$, $\sigma(z_i) = z_i$ for all i and σ determines an

automorphism on $K[x, \{z_i\}]$. It follows that $K[x, \{z_i\}]_{(x+1)} \cong K[x, \{z_i\}]_{(x)}$ and so $K[x, \{z_i\}]_{(x+1)}$ is a DVR. Hence

$$K[x, \{z_i\}, w]_{(x+1, w)} = (K[x, \{z_i\}]_{(x+1)})[w]_{(x+1, w)}$$

is a two dimensional local ring. Since $1/x$, and hence w_i for each i , is in $K[x, \{z_i\}, w]_{(x+1, w)}$, we have $(R_1)_{(x+1, w)} = K[x, \{z_i\}, w]_{(x+1, w)}$ and the claim is established.

Since $R_1[1/x, 1/(x+1)] = K[x, 1/x, w, 1/(x+1), z]$, the ideal $M_3 R_1[1/x, 1/(x+1)]$ is a rank 3 prime and so M_3 is a rank 3 prime ideal in R_1 . We have thus shown that for each $i = 1, 2, 3$, rank $M_i = i$ and $(R_1)_{M_i}$ is Noetherian. We will now use essentially Nagata's construction as previously outlined to "join" maximal ideals containing appropriate extensions of M_1 and M_2 . Let y be an indeterminate over R_1 and in $R_2 = R_1[y]$ consider the ideals $A = (x, y)$, $B = (x+1, w, y)$ and $C = (x+\beta, w, z, y)$. If $T = R_2 \setminus (A \cup B)$ then, following Nagata's construction, $R = K + (A \cap B)_T$ is a local ring with maximal ideal $P_1 = (A \cap B)_T$ of rank 3, the prime ideal $Q_1 = x(R_2)_T \cap R$ has rank one, and $P_1 \supset Q_1 \supset (0)$ is a saturated chain. Therefore $Q_1 \subset P_1$ is 1-abnormal.

We now wish to show that the ring $S_1 = R \cap (R_2)_C$ is a semilocal ring whose only maximal ideals are $P_2 = P_1 \cap S_1$ of rank 3 and $G_1 = C(R_2)_C \cap S_1$ of rank 4 and that $R = (S_1)_{P_2}$ and $(R_2)_C = (S_1)_{G_1}$. To do this we first prove that both R and $(R_2)_C$ are localizations of $K[A \cap B]$. First, we show that $R = (K[A \cap B])_{T_1}$ where $T_1 = K[A \cap B] \setminus (A \cap B)$. Clearly, since $T_1 \subseteq T$,

$$R \supseteq (K[A \cap B])_{T_1} \supseteq K + (A \cap B)_{T_1},$$

so it suffices to show that $(A \cap B)_{T_1} = (A \cap B)_T$. Considering T_1 as a multiplicative system in R_2 , let P be a prime ideal of R_2 such that $P \cap T_1 = \emptyset$. If $P \not\subseteq A \cup B$ then, since A and B are maximal ideals in R_2 , P is comaximal with $A \cap B$. Hence there exists elements $p \in P$ and $a \in A \cap B$ such that $p + a = 1$. But then $p = 1 - a \in P \cap T_1$ contrary to the choice of P . It follows that $P \subseteq A \cup B$ and so $T = R_2 \setminus (A \cup B)$ is the saturation of T_1 in R_2 . Consequently,

$$(A \cap B)_{T_1} = (A \cap B)_T \quad \text{and} \quad R = (K[A \cap B])_{T_1}.$$

Next we show that $(R_2)_C = (K[A \cap B])_{C \cap K[A \cap B]}$. To see that

$$(R_2)_C \subseteq (K[A \cap B])_{C \cap K[A \cap B]}$$

let $a/b \in (R_2)_C$, where $a \in R_2$, $b \in R_2 \setminus C$. Then

$$a/b = [ax(x+1)]/[bx(x+1)] \in (K[A \cap B])_{C \cap K[A \cap B]}.$$

Since the opposite containment is obvious, we have established the equality.

It now follows that

$$\begin{aligned} S_1 &= R \cap (R_2)_C = (K[A \cap B])_{A \cap B} \cap (K[A \cap B])_{C \cap K[A \cap B]} \\ &= (K[A \cap B])_{(K[A \cap B]) \setminus ((A \cap B) \cup (C \cap K[A \cap B]))} \end{aligned}$$

so S_1 is a semilocal ring with maximal ideals $P_2 = P_1 \cap S_1$ and $G_1 = C(R_2)_C \cap S_1$. Further, $(S_1)_{P_2} = R$ and $(S_1)_{G_1} = (R_2)_C$, so $\text{rank } P_2 = 3$ while $\text{rank } G_1 = 4$. Recall that in R , $Q_1 = x(R_2)_T \cap R$ is a rank one prime ideal such that $P_1 \supset Q_1 \supset (0)$ is a saturated chain. If we define $Q_2 = Q_1 \cap S_1$, then $Q_2 \not\subseteq G_1$ since $x(x+1) \in Q_2 \setminus G_1$. Therefore, P_2 is the only prime ideal of S_1 which properly contains Q_2 . It follows that in S_1 we have saturated chains of prime ideals as follows:

Now let $S_2 = S_1[u]$, $E = (P_2, u)$, and $G = (G_1, u)$ where u is an indeterminate over S_1 . Since P_2 is minimal over Q_2 in S_1 , $P_2[u]$ is a rank 3 prime ideal in S_2 that is minimal over $Q_2[u]$ and $\text{rank}(E/Q_2[u]) = 2$. If V is any prime ideal of S_2 that is minimal over $Q_2[u]$ then, since P_2 is the only prime ideal of S_1 properly containing Q_2 , either $V = P_2[u]$ or $V \cap S_1 = Q_2$. Hence, if $V \neq P_2[u]$ then $\text{rank } V = \text{rank } Q_2 + 1 = 2$ [3, Theorem 39]. Therefore $P_2[u]$ is the only prime ideal of S_2 with rank more than 2 that is minimal over $Q_2[u]$. Note also that since $Q_2 \not\subseteq G_1$, $Q_2[u] \not\subseteq G = (G_1, u)$.

We now perform Nagata's construction (with S_2 playing the role of the R_2 of the preliminary discussion) and "join" appropriate extensions of E and G . Specifically if we set $S = K + (E \cap G)_{T_3}$, where $T_3 = S_2 \setminus (E \cup G)$, then by the proposition in the preliminary remarks, S is the desired ring where $I_1 = (Q_2[u])_{T_3} \cap S$ and $I_2 = (E \cap G)_{T_3}$.

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