A NONLINEAR THEOREM OF ERGODIC TYPE

G. FEATHERS, J. WAYNE PACE AND W. G. DOTSON, JR.

ABSTRACT. A theorem is proved concerning the relationship between a certain collection of mappings and a fixed point for that collection of mappings. The conditions on the mappings are very similar to ones in a paper by W. F. Eberlein.

THEOREM *Let E be a Hausdorff linear topological space. Suppose $G = \{S_m\}_{m=1}^\infty$ is a collection of self-mappings of C , where C is a strongly closed subset of E , such that $I - S_m$ is demiclosed for all m . Suppose further that there is a collection of self-mappings of C , $\{C_n\}_{n=1}^\infty$ satisfying*

- (a) $\{C_n\}$ is equicontinuous on C ;
- (b) $C_n(x) \in \{S_m(x): S_m \in G\}^-$ for all n and for all $x \in C$;
- (c) Suppose $\exists x_0 \in C$ such that $C_n S_m(x_0) - C_n(x_0) \rightarrow 0$ as $n \rightarrow \infty$ and $S_m C_n(x_0) - C_n(x_0) \rightarrow 0$ as $n \rightarrow \infty$ for all $S_m \in G$.

(d) *Suppose there exist a subsequence of $\{C_n(x_0)\}_{n=1}^\infty$ such that $C_{n_i}(x_0) \rightarrow y$ (strong convergence) then*

- (1) $S_m(y) = y$ for all $S_m \in G$,
- (2) $C_n(x_0) \rightarrow y$ as $n \rightarrow \infty$.

PROOF. Since $C_n(x_0) \rightarrow y \Rightarrow C_{n_i}(x_0) \rightarrow y$ and by assumption (c) we have $(I - S_m)C_n(x_0) \rightarrow 0 \Rightarrow (I - S_m)C_{n_i}(x_0) \rightarrow 0$ so by the demiclosedness of $I - S_m$ (for all $S_m \in G$) we have $(I - S_m)y = 0$ or $y = S_m(y)$ for all $S_m \in G$. Also since $C_n(x_0) \in \{S_m(x_0): S_m \in G\}^-$ we have that $y \in \{S_m(x_0): S_m \in G\}^-$.

Now we will follow Pace's proof for part (2). Let N be an arbitrary strong neighborhood of θ . By continuity of addition on E , we may choose a strong neighborhood V of θ such that $V + V \subseteq N$. By (a) there exists a strong neighborhood W of θ such that $C_n(x) - C_n(y) \in V$ whenever $x, y \in C$ and $x - y \in W$, this holds for all n .

Since $y \in \{S_m(x_0): S_m \in G\}^-$ there exists $S_{m_1} \in G$ such that $y - S_{m_1}(x_0) \in W$. By (c) there is a n_0 such that, for all $n \geq n_0$, $C_n S_{m_1}(x_0) - C_n(x_0) \in V$. Now since $S_m(y) = y$ for all $S_m \in G$ we have $C_n(y) \in \{S_m(y): S_m \in G\}^- = \{y\}^- = \{y\}$, i.e. $C_n(y) = y$ for all n . Therefore for all $n \geq n_0$ we have

$$\begin{aligned} y - C_n(x_0) &= C_n(y) - C_n(x_0) \\ &= (C_n(y) - C_n S_{m_1}(x_0)) + (C_n S_{m_1}(x_0) - C_n(x_0)) \end{aligned}$$

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which is an element of $V + V \subseteq N$. Hence $C_n(x_0) \rightarrow y$.

REMARK 1. Let E be a uniformly convex Banach space having a weakly continuous duality mapping and let $T: E \rightarrow E$ be a nonexpansive mapping. Opial showed in [2] that $I - T^m$ is demiclosed. So if we assume that T is an asymptotically regular mapping, then by setting $S_m = T^m$ and $C_n = T^n$ all conditions of the theorem are satisfied. Thus if $T^n(x_0) \rightarrow y$ then $T^n(x_0) \rightarrow y$ and $T(y) = y$.

REMARK 2. The conditions of this theorem are very similar to the conditions in the mean ergodic theorem of W. F. Eberlein [1].

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DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NORTH CAROLINA 27607