A CHARACTERIZATION OF Z-SEPARATING ALGEBRAS

SHANKAR HEGDE

ABSTRACT. Let A be a uniformly closed point separating algebra of bounded real valued functions on a set X , containing the constant functions. A is called z -separating if whenever Z_1, Z_2 are disjoint zero sets of members of A there is some $f \in A$ with $f(Z_1) = 0$ and $f(Z_2) = 1$. We prove that A is z -separating if and only if A consists of precisely those bounded real valued functions f on X for which $f^{-1}(C)$ is a zero set of some member of A for every closed set C of real line.

Introduction. Let X be any set and A be a uniformly closed algebra of bounded real valued functions on X containing the constants and separating the points of X . We refer to A as an SW -algebra on X . Let $\mathcal{Z}[A]$ denote the lattice of zero-sets of members of A . The algebra A is called z -separating iff whenever $Z_1, Z_2 \in \mathcal{Z}[A]$ with $Z_1 \cap Z_2 = \emptyset$ there exists $f \in A$ such that $f(Z_1) = 0$ and $f(Z_2) = 1$. The z -separating algebras were introduced and studied by Kirk in [8]. For a lattice $\mathcal{L}$ of subsets of X (we assume that $\mathcal{L}$ contains $\emptyset$ and X), a real valued function f on X is said to be $\mathcal{L}$ -continuous if $f^{-1}(C) \in \mathcal{L}$ for every closed set C of R , the real line. We denote the algebra of all bounded $\mathcal{L}$ -continuous functions on X by $C_b(\mathcal{L})$. The purpose of this note is to prove that A is z -separating iff $A = C_b(\mathcal{Z}[A])$.

For a Tychonoff space X , we denote the algebra of all real valued continuous functions on X by $C(X)$ and all the bounded ones by $C_b(X)$. Various classes of subalgebras of $C(X)$ were studied by Hager [5], [6], [7], Frolik [4] and others. Since an SW -algebra A on a set X can always be considered as a subalgebra of $C_b(X)$ with a suitable topology on X , our study characterizes a class of subalgebras of $C_b(X)$. Note that in general a z -separating algebra A need not be $C_b(X)$ (see Theorem 3.12 and Example 3.5 of Kirk). If $A = C_b(X)$, then we always have $A = C_b(\mathcal{Z}[A])$. The present note proves that this property characterizes the z -separating algebras. More generally, we deduce that A is z -separating iff there exists a lattice $\mathcal{L}$ of subsets of X which is closed under countable intersections, such that $A = C_b(\mathcal{L})$. In [6] Hager noted that the uniformly closed cbq (closed under bounded quotients) vector lattices V of bounded real valued functions on a set X can be characterized as the families V for which $V = C_b(\mathcal{Z}[V])$.

Received by the editors October 31, 1977 and, in revised form, April 6, 1978.

AMS (MOS) subject classifications (1970). Primary 46E25; Secondary 46J10, 46J35.

Key words and phrases. z -separating algebra, Wallman compact space, $\mathcal{L}$ -continuity and $\mathcal{L}$ -uniform continuity.

0002-9939/79/0000-0009/\$02.25
 © 1979 American Mathematical Society

Combining this remark with our result it follows that z -separating algebras are precisely the SW -algebras which are closed under bounded quotients.

1. Preliminaries. Given an SW -algebra A on a set X , there exists, as a consequence of the Stone-Weierstrass theorem, a compact Hausdorff space X_A such that X can be embedded into X_A as a dense subspace and A is isomorphic as a Banach lattice to $C_b(X_A)$ [2, p. 276]. In fact, X_A is the structure space of A , that is, the set of all nonzero real homomorphisms of A topologized with the weak star topology. Thus X_A is a Hausdorff compactification of X and A consists of precisely those continuous functions on X which can be continuously extended to X_A . If Y and Z are two compactifications of X , we write $Y = Z$ if there exists a homeomorphism of Y onto Z leaving X pointwise fixed.

The idea of the proof of the main theorem is to show that X_A is a "Wallman-type compactification" of X and then apply Frink's extension theorem. A lattice $\mathcal{L}$ of subsets of a set X is called *strongly normal* if it is (i) *separating*: given x and y in X , $x \neq y$, there exists an $L \in \mathcal{L}$ such that $x \in L$ and $y \notin L$; (ii) *disjunctive*: for any $L \in \mathcal{L}$ and $x \notin L$, there exists $M \in \mathcal{L}$ such that $x \in M$ and $L \cap M = \emptyset$ and (iii) *normal*: whenever $L, M \in \mathcal{L}$ with $L \cap M = \emptyset$, there exists $U, V \in \mathcal{L}$ such that $L \subset X - U$, $M \subset X - V$ and $U \cup V = X$. Given a strongly normal lattice $\mathcal{L}$ of subsets of X , we topologize the set $\omega(\mathcal{L})$ of all $\mathcal{L}$ -ultrafilters by taking a base for the closed sets, the sets of the form $\bar{L} = \{\mathcal{U} \in \omega(\mathcal{L}) : L \in \mathcal{U}\}$ where $L \in \mathcal{L}$. Then $\omega(\mathcal{L})$ is a compact Hausdorff space and the mapping $x \rightarrow \mathcal{U}_x$ where $\mathcal{U}_x$ is the $\mathcal{L}$ -ultrafilter consisting of all $L \in \mathcal{L}$ for which $x \in L$, is an embedding of X into $\omega(\mathcal{L})$ and X , treated as a subspace of $\omega(\mathcal{L})$ is dense in it (see Brooks [1]). We refer to $\omega(\mathcal{L})$ as the Wallman compact space relative to $\mathcal{L}$. Given a lattice $\mathcal{L}$ of subsets of X , a real valued function f on X is said to be $\mathcal{L}$ -uniformly continuous if for each $\varepsilon > 0$, there exists a finite family $\{L_1, \dots, L_n\}$ of members of $\mathcal{L}$ such that $\bigcap_{i=1}^n L_i = \emptyset$ and the oscillation of f on each of the sets $X - L_i$ is less than ε . The following extension theorem is proved in [3].

THEOREM 1.1 (O. FRINK). *Let $\mathcal{L}$ be a strongly normal lattice of subsets of X . For each real valued function f on X , there exists an extension $\bar{f}$ of f to $\omega(\mathcal{L})$ such that $\bar{f} \in C(\omega(\mathcal{L}))$ iff f is $\mathcal{L}$ -uniformly continuous.*

We note that a $\mathcal{L}$ -uniformly continuous function is continuous on X but need not be $\mathcal{L}$ -continuous on X .

2. The results. In this section we prove the theorem stated in the introduction and deduce some interesting corollaries. First we prove a few lemmas.

LEMMA 2.1. *Let A be an SW -algebra on a set X . Then (i) $\mathcal{Z}[A]$ is a strongly normal lattice on X . (ii) $X_A = \omega(\mathcal{Z}[A])$ iff A is z -separating.*

PROOF. Equip X with the relative topology inherited from the topology of

X_A . Since $\mathcal{Z}[A]$ is the trace of zero sets of X_A on X , the results follow from Theorems 5.3 (b) and 5.2 of Hager [5].

REMARKS. If T is a compactification of a Tychonoff space X and $\mathcal{Z}_T$ denotes the trace of the zero-sets of T on X , then Hager [5] has proved that $\mathcal{Z}_T$ is a normal base on X and $\omega(\mathcal{Z}_T)$ coincides with the structure space of $A(T)$, where $A(T)$ is the smallest uniformly closed, inverse closed subalgebra of $C(X)$ containing all those members which are continuously extendible to T . Then by Lemma 2.1 it follows that A is z -separating iff X_A is the structure space of the inverse closed algebra $A(X_A)$. Thus in this case A consists of precisely the bounded members of $A(X_A)$.

LEMMA 2.2 *For any lattice $\mathcal{L}$ of subsets of a set X , every bounded $\mathcal{L}$ -continuous function on X is $\mathcal{L}$ -uniformly continuous.*

PROOF. Let $f \in C_b(\mathcal{L})$. Let $a \leq f(x) \leq b$ for every $x \in X$. Given an arbitrary $\varepsilon > 0$, let $0 < \delta < \varepsilon/2$. Choose $a_1, a_2, \dots, a_n$ in $[a, b]$ such that $[a, b] \subset \bigcup_{i=1}^n (a_i - \delta, a_i + \delta)$. Then for $i = 1, \dots, n$, $L_i = f^{-1}(-\infty, a_i - \delta] \cup f^{-1}[a_i + \delta, \infty) \in \mathcal{L}$. Now it is readily verified that $\bigcap_{i=1}^n L_i = \emptyset$ and the oscillation of f on each of the sets $X - L_i$ is less than ε . Thus $f \in C_u(\mathcal{L})$, where $C_u(\mathcal{L})$ denotes the set of all $\mathcal{L}$ -uniformly continuous functions on X .

A family of sets is called a *delta family* if it is closed under countable intersections. Next we prove

LEMMA 2.3. *If $\mathcal{L}$ is a strongly normal delta lattice of subsets of a set X , then $C_b(\mathcal{L}) = C_u(\mathcal{L})$.*

PROOF. In view of Lemma 2.2, it is enough to show that $C_u(\mathcal{L}) \subset C_b(\mathcal{L})$. First we note that for a delta lattice $\mathcal{L}$, $f \in C_b(\mathcal{L})$ iff $f^{-1}(-\infty, a]$ and $f^{-1}[b, \infty) \in \mathcal{L}$ for all real numbers a and b .

Recall the construction of $\omega(\mathcal{L})$. The family $\bar{\mathcal{L}} = \{\bar{L} : L \in \mathcal{L}\}$ where $\bar{L}$ consists of $\mathcal{L}$ -ultrafilters of which L is a member, is a base for the closed sets of $\omega(\mathcal{L})$. Also it is easy to see that $\bar{\mathcal{L}}$ is a strongly normal lattice of subsets of $\omega(\mathcal{L})$. Let $\rho(\bar{\mathcal{L}})$ denote the delta lattice consisting of countable intersections of members of $\bar{\mathcal{L}}$. One can check that $C_b(\rho(\bar{\mathcal{L}}))$ is an *SW*-algebra on the compact Hausdorff space $\omega(\mathcal{L})$ and hence by the Stone-Weierstrass theorem, $C_b(\rho(\bar{\mathcal{L}})) = C(\omega(\mathcal{L}))$.

Let $f \in C_u(\mathcal{L})$. Let $\bar{f}$ denote the unique continuous extension of f to $\omega(\mathcal{L})$. Let $a \in \mathbb{R}$. Then $\bar{f}^{-1}(-\infty, a] = \bigcap_{n=1}^{\infty} \bar{B}_n$ for some sequence $\{B_n\}$ in $\mathcal{L}$. Hence $f^{-1}(-\infty, a] = X \cap \bar{f}^{-1}(-\infty, a] = \bigcap_{n=1}^{\infty} (X \cap \bar{B}_n) = \bigcap_{n=1}^{\infty} B_n$ since each $B_n \in \mathcal{L}$ is closed in X . Now $\mathcal{L}$ being a delta lattice, $f^{-1}(-\infty, a] \in \mathcal{L}$. Similarly it follows that $f^{-1}[b, \infty) \in \mathcal{L}$ for every $b \in \mathbb{R}$. Thus $f \in C_b(\mathcal{L})$. This completes the proof.

In particular, given any *SW*-algebra A on a set X we always have $C_b(\mathcal{Z}[A]) = C_u(\mathcal{Z}[A])$.

THEOREM 2.4. *Let A be an SW-algebra on a set X . Then A is z -separating iff $A = C_b(\mathcal{Z}[A])$.*

PROOF. Suppose A is z -separating. Let $f \in C_b(\mathcal{Z}[A])$. By Lemma 2.3 and Theorem 1.1, f can be extended to be a continuous function on $\omega(\mathcal{Z}[A])$. By Lemma 2.1 $X_A = \omega(\mathcal{Z}[A])$ so that there is an isomorphism of $C(\omega(\mathcal{Z}[A]))$ onto $C(X_A)$ such that the corresponding elements under this isomorphism agree on X . Hence f has a continuous extension $\tilde{f} \in C(X_A)$. Since A consists of precisely the restrictions of members of $C(X_A)$ to X , $f \in A$. Thus $C_b(\mathcal{Z}[A]) \subset A$.

On the other hand we note that for any SW-algebra A we have $A \subset C_b(\mathcal{Z}[A])$. In fact, since A contains the constants and is closed under the lattice operations, for every pair of real numbers a and b and every $f \in A$, $g = \max(f, a) - a \in A$ and $h = b - \min(b, f) \in A$. Hence $f^{-1}(-\infty, a] = Z(g) \in \mathcal{Z}[A]$, $f^{-1}[b, \infty) = Z(h) \in \mathcal{Z}[A]$. This proves that $A = C_b(\mathcal{Z}[A])$.

Conversely suppose $A = C_b(\mathcal{Z}[A])$. As is well known $C_b(\mathcal{Z}[A])$ is closed under bounded quotients (see e.g. [7]). Let $Z_1 = Z(f)$, $Z_2 = Z(g)$ be disjoint members of $\mathcal{Z}[A]$ where $f, g \in A$. Let $h = |f|(|f| + |g|)^{-1}$. Then $0 \leq h \leq 1$. Hence $h \in A$. Furthermore $h(Z_1) = 0$ and $h(Z_2) = 1$, proving that A is z -separating.

As immediate consequences we have

COROLLARY 2.5. *For any SW-algebra A on a set X , $C_b(\mathcal{Z}[A])$ is the smallest z -separating algebra containing A .*

COROLLARY 2.6 (KIRK). *Let X be a Tychonoff space and A be a z -separating subalgebra of $C_b(X)$. Then $A = C_b(X)$ iff $\mathcal{Z}[A] = \mathcal{Z}[C_b(X)]$.*

We further deduce

COROLLARY 2.7. *Let A be an SW-algebra on a set X . Then A is z -separating iff there exists a delta lattice $\mathcal{L}$ of subsets of X such that $A = C_b(\mathcal{L})$.*

PROOF. For a delta lattice $\mathcal{L}$ of subsets of X , one can readily check that $C_b(\mathcal{L}) = C_b(\mathcal{Z}[C_b(\mathcal{L})])$. Thus $A = C_b(\mathcal{L})$ implies $A = C_b(\mathcal{Z}[A])$. For the converse take $\mathcal{L} = \mathcal{Z}[A]$.

REMARKS. In [1] Brooks pointed out that if $\mathcal{L}$ is a strongly normal lattice of subsets of X , then the finite covers of X from the family of $\mathcal{L}$ -complements generate a separated uniform structure on X and $C_u(\mathcal{L})$ is precisely the family of bounded real valued functions on X which are uniformly continuous relative to this structure. In the light of this remark compare our results with those of Frolik [4] on uniform spaces.

For a lattice $\mathcal{L}$ of subsets of X , let $MR(\mathcal{L})$ denote the space of all finite, finitely additive, $\mathcal{L}$ -regular, real valued set functions on the field of subsets of X generated by $\mathcal{L}$. $MR(\mathcal{L})$ is said to *represent* A^* , the Banach dual of A , if there exists a (unique) Banach lattice isomorphism I from A^* onto $MR(\mathcal{L})$

such that for each nonnegative $\psi \in A^*$, $I\psi(W) = \inf\{\psi(f): f \in A, \mathcal{K}_W \leq f\}$ for every $W \in \mathcal{L}$. Kirk [8] has proved that $MR(\mathcal{Z}[A])$ represents A^* iff A is z -separating. Thus we obtain

COROLLARY 2.8. $MR(\mathcal{Z}[A])$ represents A^* iff $A = C_b(\mathcal{Z}[A])$.

The author is grateful to Professor P. Cuttle for helpful discussions and wishes to thank the referee for his comments concerning the works of Hager and Frolik.

REFERENCES

1. R. M. Brooks, *On Wallman compactifications*, Fund. Math. **60** (1967), 157–173.
2. N. Dunford and J. T. Schwartz, *Linear operators* Part I, Interscience, New York, 1958.
3. O. Frink, *Compactifications and seminormal spaces*, Amer. J. Math. **86** (1964), 602–607.
4. Z. Frolik, *Three uniformities associated with uniformly continuous functions*, Symposia Math. **17** (1976), 69–80.
5. A. W. Hager, *On inverse closed subalgebras of $C(X)$* , Proc. London Math. Soc. **19** (1969), 233–257.
6. ———, *Real-valued functions on Alexandroff (zero-set) spaces*, Comment. Math. Univ. Carolinae **16** (1975), 755–769.
7. ———, *A class of function algebras (and compactifications and uniform spaces)*, Symposia Math. **17** (1976), 11–23.
8. R. B. Kirk, *Algebras of bounded real valued functions*. I, II, Nederl. Akad. Wetensh. Proc. Ser. A **75** = Indag. Math. **34** (1972), 443–463.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SASKATCHEWAN, SASKATOON, SASKATCHEWAN, CANADA S7N 0W0

Current address: Department of Mathematical Sciences, Northern Illinois University, DeKalb, Illinois 60115