UNIVERSAL LUSIN MEASURABILITY AND SUBFAMILY SUMMABLE FAMILIES IN ABELIAN TOPOLOGICAL GROUPS

WILLIAM H. GRAVES

ABSTRACT. It is proved that if G is a Hausdorff abelian topological group with respect to topologies $\alpha \subseteq \beta$ such that β is complete and the identity map of (G, α) onto (G, β) is universally Lusin measurable, then the subfamily summable families are the same for α and β .

0. Introduction. Kalton's remarkable paper [5] was the forerunner among other recent entrants in a newly directed study of the Orlicz-Pettis theorem. The papers [5], [6] of Kalton, [3], [4] of Drewnowski, [2] of Christensen, [1] of Andersen and Christensen, and [7], [8], [9] of Labuda are part of a shift from weak-implies-how-strong studies to strong-is-implied-by-how-weak studies. In the last of these papers is established a result conjectured by Thomas in [12]. The purpose of this paper is to give a very simple proof, based on nothing deeper than Fubini's theorem, of a variation on the conjecture of Thomas. It is here proved that if G is a Hausdorff abelian topological group with respect to two topologies α, β such that $\alpha \subseteq \beta$, β is complete, and the identity map $\mathcal{I}: (G, \alpha) \rightarrow (G, \beta)$ is universally Lusin measurable, then the subfamily summable families in G are the same for α and β . Labuda's version of this in [9] is set in the category of Hausdorff topological vector spaces, assumes universal measurability of $\mathcal{I}$ rather than universal Lusin measurability, draws upon deeper results than does the version proved herein, and may be obtained in generalized form from the main result here.

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1. Preliminaries. Let S be an infinite set. $\mathcal{P}(S) = \{0, 1\}^S$ is the topological product of S copies of the discrete space $\{0, 1\}$ and so is a compact Hausdorff space. The power set of S will be identified throughout with $\mathcal{P}(S)$, and no distinction will be made between a set and its characteristic function.

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It is with this understanding that set-theoretic terms such as $a \cup b$ and $a \setminus b$ are defined for $a, b \in \mathcal{P}(S)$. For $s \in S$, e_s is the characteristic function of $\{s\}$.

$G \in \mathcal{G} \mathcal{R} \mathcal{P}$ means that G is a Hausdorff abelian topological group. $\Phi: \mathcal{P}(S) \rightarrow G$ is *additive* if $\Phi(a \cup b) = \Phi(a) + \Phi(b)$ for $a, b \in \mathcal{P}(S)$ such that $a \cap b = 0$. For additive Φ and $s \in S$, let $\Phi_s = \Phi(e_s)$. If additive Φ is continuous, then $\{\Phi_s: s \in S\} = \{\Phi_s: s \in S\}$ is a *subfamily summable family* in G in that $\{\Phi_s: s \in S'\}$ is summable for every $S' \subseteq S$ ($\Phi(a) = \sum_{s \in a} \Phi_s$ for $a \in \mathcal{P}(S)$). Conversely, if $\{\Phi_s\}$ is a subfamily summable family in G , then $\Phi(a) = \sum_{s \in a} \Phi_s$ defines an additive, continuous map $\Phi: \mathcal{P}(S) \rightarrow G$. With these conventions of identification, this equivalence can be condensed.

1.1. PROPOSITION. *Let $\Phi: \mathcal{P}(S) \rightarrow G \in \mathcal{G} \mathcal{R} \mathcal{P}$ be additive. Then Φ is continuous if and only if $\{\Phi_s\}$ is subfamily summable and $\Phi(a) = \sum_{s \in a} \Phi_s$ for each $a \in \mathcal{P}(S)$.*

It is well known that in the special case $S = \mathbb{N}$, the set of positive integers, a sequence (Φ_n) is subfamily summable if and only if it is *subseries convergent* in that $\sum \Phi_{n_i}$ converges for every subsequence (Φ_{n_i}) of (Φ_n) . So 1.1 has special form in this case.

1.2. PROPOSITION. *Let $\Phi: \mathcal{P}(\mathbb{N}) \rightarrow G \in \mathcal{G} \mathcal{R} \mathcal{P}$ be additive. Then Φ is continuous if and only if (Φ_n) is subseries convergent and $\Phi(a) = \sum_{n \in a} \Phi_n$ for each $a \in \mathcal{P}(\mathbb{N})$.*

Additive $\Phi: \mathcal{P}(S) \rightarrow G \in \mathcal{G} \mathcal{R} \mathcal{P}$ is *exhaustive* if $\Phi(a_n) \rightarrow 0$ for every sequence (a_n) of pairwise disjoint members of $\mathcal{P}(S)$.

All topological spaces will be Hausdorff. For such X , let $\mathcal{B}_X$ be the σ -algebra of all Borel subsets of X (generated by the open subsets). $\mu \in \mathcal{R}(X)$ means that μ is a finite nonnegative *Radon measure* on X ; that is, $\mu: \mathcal{B}_X \rightarrow \mathbb{R}^+$ is countably additive and both open regular and compact regular. So when X is compact, $\mathcal{R}(X)$ consists of all regular Borel measures in the usual sense. For $\mu \in \mathcal{R}(X)$, $M \subseteq X$ is μ -*measurable* if $B_1 \subseteq M \subseteq B_2$ for some $B_1, B_2 \in \mathcal{B}_X$ such that $\mu(B_2) - \mu(B_1) = 0$, and $\mathcal{M}_\mu$ is the σ -algebra of all such M . For $\mu \in \mathcal{R}(X)$, $f: X \rightarrow Y$ is μ -*measurable* if $f^{-1}(\mathcal{B}_Y) \subseteq \mathcal{M}_\mu$ and is *Lusin μ -measurable* if for $\varepsilon > 0$ and compact $K \subseteq X$ there is compact $K_\varepsilon \subseteq K$ such that $\mu(K) - \mu(K_\varepsilon) < \varepsilon$ and f is continuous on K_ε . f is *universally measurable* if μ -measurable for all $\mu \in \mathcal{R}(X)$ and is *universally Lusin measurable* if Lusin μ -measurable for all $\mu \in \mathcal{R}(X)$.

The next two results are but special cases of more general results available in [11]. Proofs are included here in order that the simplicity of the results in the next section will be evident.

1.3. LEMMA. *For compact $K \subseteq X$, continuous $f: K \rightarrow Y$, and $\mu \in \mathcal{R}(X)$, define $f(\mu): \mathcal{B}_Y \rightarrow \mathbb{R}$ by $f(\mu)(B) = \mu(f^{-1}(B))$, $B \in \mathcal{B}_Y$. Then $f(\mu) \in \mathcal{R}(Y)$.*

PROOF. $f(\mu)$ is well defined since $f^{-1}(\mathcal{B}_Y) \subseteq \mathcal{B}_K \subseteq \mathcal{B}_X$ and is clearly

countably additive. Consider $B \in \mathfrak{B}_Y$ and $\varepsilon > 0$. Pick compact $K_0 \subseteq f^{-1}(B)$ such that $\mu(f^{-1}(B)) - \mu(K_0) < \varepsilon$. Then $f(K_0)$ is compact, $f(K_0) \subseteq B$, and $f(\mu)(B) - f(\mu)(f(K_0)) < \varepsilon$, proving that $f(\mu)$ is compact regular. Choose open $V \supseteq f^{-1}(B)$ such that $\mu(V) - \mu(f^{-1}(B)) < \varepsilon$. Then $f(K \setminus V \cap K)$ is compact, and $V_1 = Y \setminus f(K \setminus V \cap K)$ is open as is $f^{-1}(V_1)$. But $f^{-1}(B) \subseteq f^{-1}(V_1) \subseteq V \cap K$ and $f(\mu)(V_1) - f(\mu)(B) < \varepsilon$, proving that $f(\mu)$ is open regular.

1.4. PROPOSITION. *If $f: X \rightarrow Y$ is continuous and $g: Y \rightarrow Z$ is universally Lusin measurable, then $g \circ f$ is universally Lusin measurable.*

PROOF. Consider $\mu \in \mathfrak{R}(X)$, compact K , and $\varepsilon > 0$. Let f_K be the restriction of f to K . Then $f_K(\mu) \in \mathfrak{R}(Y)$ by 1.3. Choose compact $C_\varepsilon \subseteq f(K)$ such that $f_K(\mu)(f(K)) - f_K(\mu)(C_\varepsilon) < \varepsilon$ and g is continuous on C_ε . Let $K_\varepsilon = f_K^{-1}(C_\varepsilon)$. Then K_ε is compact, $g \circ f$ is continuous on K_ε , and $\mu(K) - \mu(K_\varepsilon) < \varepsilon$.

2. Key lemma and main results. Let μ be the Radon measure defined on the discrete space $\{0, 1\}$ by $\mu(\{0\}) = \frac{1}{2} = \mu(\{1\})$. Let λ be the associated product measure on $\{0, 1\}^{\mathbb{N}} = \mathfrak{P}(\mathbb{N})$, also a Radon measure. Let $n \in \mathbb{N}$. Let $\mathfrak{P}(\mathbb{N} \setminus n) = \{0, 1\}^{\mathbb{N} \setminus \{n\}}$, and let λ_n be the product Radon measure on $\mathfrak{P}(\mathbb{N} \setminus n)$ determined by μ . Then $\mathfrak{P}(\mathbb{N}) = \{0, 1\} \times \mathfrak{P}(\mathbb{N} \setminus n)$, and for a Borel set $A \subseteq \mathfrak{P}(\mathbb{N})$, Fubini's theorem gives

$$\lambda(A) = \int \mu(A_c) d\lambda_n(c)$$

where for $c \in \mathfrak{P}(\mathbb{N} \setminus n)$,

$$A_c = \{k \in \{0, 1\} : (k, c) \in A\}$$

is the c -section of A .

2.1. LEMMA. *If $\Phi: \mathfrak{P}(\mathbb{N}) \rightarrow G \in \mathfrak{G} \mathfrak{R} \mathfrak{P}$ is additive and Lusin λ -measurable, then $\Phi_n \rightarrow 0$.*

PROOF. Choose compact $K \subseteq \mathfrak{P}(\mathbb{N})$ such that $\lambda(\mathfrak{P}(\mathbb{N}) \setminus K) < \frac{1}{2}$ and Φ is continuous on K . For $n \in \mathbb{N}$, let

$$K_n = \{a \in K : a \cup e_n, a \setminus e_n \in K\},$$

and let $A = K \setminus K_n$. A is a Borel set since K_n is compact. For $c \in \mathfrak{P}(\mathbb{N} \setminus n)$, the c -section A_c has at most one element, and so $\lambda(A) \leq \frac{1}{2}$ by Fubini's theorem. But

$$\frac{1}{2} < \lambda(K) = \lambda(A) + \lambda(K_n);$$

whence $\lambda(K_n) > 0$ proving that $K_n \neq \emptyset$ for each $n \in \mathbb{N}$. Choose $a_n \in K_n$ for each $n \in \mathbb{N}$. If $\lim \Phi_n \neq 0$, there is a 0-neighborhood $\mathcal{U}$ in G and a subsequence $\Phi_{n_k} \notin \mathcal{U}$. Since K is compact, there exist $a \in K$ and a subsequence $(a_{n_{k_i}})$ of (a_{n_k}) converging to a . But $a_n \cup e_n = (a_n \setminus e_n) \cup e_n$ is a

disjoint union for all n , both

$$a_{n_{k_i}} \cup e_{n_{k_i}} \rightarrow a \quad \text{and} \quad a_{n_{k_i}} \setminus e_{n_{k_i}} \rightarrow a$$

as $i \rightarrow \infty$, and for each i ,

$$a_{n_{k_i}} \cup e_{n_{k_i}}, \quad a_{n_{k_i}} \setminus e_{n_{k_i}} \in K$$

by definition of the a_n 's. So by continuity of Φ on K ,

$$\Phi_{n_{k_i}} = \Phi(a_{n_{k_i}} \cup e_{n_{k_i}}) - \Phi(a_{n_{k_i}} \setminus e_{n_{k_i}}) \rightarrow 0$$

contradicting $\Phi_{n_{k_i}} \notin \mathcal{U}$.

2.2. THEOREM. *If $\Phi: \mathcal{P}(S) \rightarrow G \in \mathcal{GRLP}$ is additive and universally Lusin measurable, then Φ is exhaustive.*

PROOF. If not, then $\Phi(a_n) \notin \mathcal{U}$ for some sequence (a_n) of pairwise disjoint subsets of S and some 0-neighborhood $\mathcal{U}$ in G . Define a continuous map $f: \mathcal{P}(\mathbb{N}) \rightarrow \mathcal{P}(S)$ by $f(b) = \bigcup_{n \in b} a_n$ and let $\Psi = \Phi \circ f$. Then Ψ is universally Lusin measurable by 1.4. So $\Phi(a_n) = \Psi_n \rightarrow 0$ by 2.1, a contradiction.

2.3. COROLLARY. *For $(G, \alpha), (G, \beta) \in \mathcal{GRLP}$, assume that $\alpha \subseteq \beta$, that (G, β) is complete, and that the identity map $\mathcal{I}: (G, \alpha) \rightarrow (G, \beta)$ is universally Lusin measurable. Then the subfamily summable families are the same for (G, α) and (G, β) .*

PROOF. It suffices to prove that an α -subfamily summable family $\{\Phi_s\}$ is β -subfamily summable. Let $\Phi: \mathcal{P}(S) \rightarrow (G, \alpha)$ be the associated continuous, additive function. Let $\Phi_\beta = \mathcal{I} \circ \Phi$. Then Φ_β is universally Lusin measurable by 1.4 and so is β -exhaustive by 2.2. Since β is complete, $\alpha \subseteq \beta$, and $\{\Phi_s\}$ is α -subfamily summable, it follows that it is β -subfamily summable.

It is clear from the above proof that the special case $S = \mathbb{N}$ of 2.3 requires only β -sequential completeness.

2.4. COROLLARY. *For $(G, \alpha), (G, \beta) \in \mathcal{GRLP}$, assume that $\alpha \subseteq \beta$, that (G, β) is sequentially complete, and that the identity map $\mathcal{I}: (G, \alpha) \rightarrow (G, \beta)$ is universally Lusin measurable. Then the subseries convergent series are the same for (G, α) and (G, β) .*

The next corollary of 2.2 is a variant of a well-known theorem of Banach.

2.5. COROLLARY. *Let $G \in \mathcal{GRLP}$ have topology determined by a complete, invariant metric. If a homomorphism $f: G \rightarrow G' \in \mathcal{GRLP}$ is universally Lusin measurable, then it is continuous.*

PROOF (LABUDA). If f is not continuous, then $f(g_n) \notin \mathcal{U}$ for some 0-neighborhood $\mathcal{U}$ in G' and some sequence (g_n) in G such that $g_n \rightarrow 0$. By completeness and invariance of the metric on G , there is a subsequence (g_{n_i}) which is subseries convergent and hence determines a continuous, additive map $\Phi: \mathcal{P}(\mathbb{N}) \rightarrow G$ such that $\Phi_i = g_{n_i}$. Then $\Psi = f \circ \Phi$ is additive and exhaustive by 1.4 and 2.2. Thus, $f(g_{n_i}) = \Psi_i \rightarrow 0$, a contradiction.

3. Further results and connections with other work. Except for the reference to Fubini's theorem in the proof of 2.2, this paper has been self-contained and erected on a very elementary base up to this point. When restricted to the category of topological vector spaces, 2.2–2.5 follow from Labuda's deeper work in [9] since universal measurability follows from universal Lusin measurability. In order to generate Kalton's result in [5] and results of Labuda in [9] from 2.2–2.5, it is necessary to invoke measurability considerations deeper than those in §1. These measurability results are listed below.

Recall that X is K -analytic if it is the continuous image of a $K_{\sigma\delta}$ subspace of some space and that X is Suslin if it is the continuous image of a separable, complete metric space. A locally convex topological vector space is Fréchet if its topology is that of a complete, invariant metric.

3.1. THEOREM. *For $\mu \in \mathcal{R}(X)$, if $f: X \rightarrow Y$ is μ -Lusin measurable, then it is μ -measurable. The converse is true if Y is Suslin.*

PROOF. See [11, Theorem 5, p. 26 and Theorem 14, p. 129].

3.2. THEOREM. *If topology α on X is coarser than topology β and (X, β) is K -analytic, then the identity map $\mathcal{G}: (X, \alpha) \rightarrow (X, \beta)$ is universally Lusin measurable.*

PROOF. See [11, Corollary 1, p. 126].

3.3. THEOREM. *If E is a Fréchet space, then the identity map $\mathcal{G}: (E, \text{weak}) \rightarrow E$ is universally Lusin measurable.*

PROOF. See [11, Theorem 3, p. 162].

3.4. THEOREM. *Let $G \in \mathcal{G} \mathcal{P} \mathcal{R}$ be complete and K -analytic. The subfamily summable families coincide for all Hausdorff topologies coarser than the original one and compatible with the group structure.*

PROOF. Apply 2.3 and 3.2. For Kalton's version, see [5, Theorem 3].

3.5. THEOREM. *If $\Phi: \mathcal{P}(S) \rightarrow G \in \mathcal{G} \mathcal{R} \mathcal{P}$ is additive and universally measurable, then Φ is exhaustive.*

PROOF (LABUDA). G may be embedded in a product, $\prod G_\gamma$, of complete metrizable abelian groups. For each γ , let Φ_γ be the composition of Φ with the projection of G on G_γ . Then Φ_γ is universally measurable and, by a result of Sazonov [10, Corollaries 1 and 2 of Theorem 11], is even μ -essentially separably valued for every Radon measure μ on $\mathcal{P}(S)$. So by 3.1, each Φ_γ is μ -Lusin measurable for all such μ , and the result now follows from 2.2.

Now Labuda's main result in [9], generalized to groups, follows from 3.5 just as 2.3 follows from 2.2 since $\mathcal{G} \circ \Phi$ is universally measurable when Φ is continuous and $\mathcal{G}$ is universally measurable, [9, Proposition 1.3].

3.6. COROLLARY. For $(G, \alpha), (G, \beta) \in \mathcal{GRP}$, assume that $\alpha \subseteq \beta$, that (G, β) is complete, and that the identity map $\mathcal{I}: (G, \alpha) \rightarrow (G, \beta)$ is universally measurable. Then the subfamily summable families are the same for (G, α) and (G, β) .

3.7. THEOREM. If E is a Fréchet space, then the subfamily summable families (in particular, the subseries convergent series) are the same for the weak and the original (= strong) topology.

PROOF. This follows from 2.3 and 3.3 and is the classical Orlicz-Pettis theorem.

A glance at its proof reveals that 2.5 can be strengthened via 3.6.

3.8. THEOREM. Let $G \in \mathcal{GRP}$ have topology determined by a complete, invariant metric. If a homomorphism $f: G \rightarrow G' \in \mathcal{GRP}$ is universally measurable, then it is continuous.

REFERENCES

1. N. J. M. Andersen and J. P. R. Christensen, *Some results on Borel structures with applications to subseries convergence in abelian topological groups*, Israel J. Math. **15** (1973), 414–420.
2. J. P. R. Christensen, *Borel structures and a topological zero-one law*, Math. Scand. **29** (1971), 245–255.
3. L. Drewnowski, *On the Orlicz Pettis type theorems of Kalton*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **21** (1973), 515–518.
4. ———, *Another note on Kalton's theorems*, Studia Math. **52** (1975), 233–237.
5. N. J. Kalton, *Subseries convergence in topological groups and vector spaces*, Israel J. Math. **10** (1971), 402–412.
6. ———, *Topologies on Riesz groups and applications to measure theory*, Proc. London Math. Soc. (3) **28** (1974), 253–273.
7. I. Labuda, *A generalization of Kalton's theorem*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **21** (1973), 509–510.
8. ———, *Discrete copies of rings of sets in groups and Orlicz-Pettis theorem* (preprint).
9. ———, *Universal measurability and summable families in tvs* (preprint).
10. V. V. Sazonov, *On perfect measures*, Amer. Math. Soc. Transl. (2) **48** (1965), 229–254.
11. L. Schwartz, *Radon measures on arbitrary topological spaces and cylindrical measures*, Oxford Univ. Press, New York and London, 1973.
12. E. Thomas, *The Lebesgue-Nikodym theorem for vector-valued Radon measures*, Mem. Amer. Math. Soc. No. 139 (1974).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH CAROLINA, CHAPEL HILL, NORTH CAROLINA 27514