A VOLTERRA EQUATION WITH NONINTEGRABLE RESOLVENT

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ABSTRACT. A counterexample, which shows that the resolvent associated with a nonnegative, nonincreasing function need not be integrable, is constructed.

For a locally integrable function, a , on $R_+ = [0, \infty)$ the resolvent (kernel) is defined to be the unique solution, r , of

$$r(t) + \int_0^t a(t-s)r(s) ds = a(t), \quad t \in R_+, \quad (1)$$

The importance of the resolvent derives from the fact that the solution of the linear Volterra equation

$$x(t) + \int_0^t a(t-s)x(s) ds = f(t), \quad t \in R_+, \quad (2)$$

is given by

$$x(t) = f(t) - \int_0^t r(t-s)f(s) ds, \quad t \in R_+. \quad (3)$$

Equation (3) shows that if one knows that $r \in L^1(R_+)$ then one can draw a number of conclusions concerning the asymptotic behaviour of the solution x from the corresponding behaviour of the function f . Various conditions implying that $r \in L^1(R_+)$ have been given in e.g. [1]–[3], [5]–[7]. If on the other hand $r \notin L^1(R_+)$ then the asymptotic behaviour of the solution x can differ from that of the function f , see [8]. In [4] the question was raised whether it is sufficient for the integrability of r to assume that a is nonnegative, nonincreasing with $a(0) < \infty$. Here we show that this is not the case and we prove a slightly stronger result.

THEOREM. *The condition*

$$a: [0, \infty) \rightarrow [0, \infty), \quad a \text{ nonincreasing}, \quad (4)$$

does not imply that the mapping: $f \rightarrow \int_0^t r(t-s)f(s) ds$, where r is the resolvent associated with a , maps $L^2(R_+)$ into itself.

PROOF. For any function a that satisfies (4) the Laplace transform defined

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$$\hat{a}(s) = \int_0^\infty e^{-st} a(t) dt \quad (5)$$

is in fact continuous in the set $\{s | \operatorname{Re} s \geq 0, s \neq 0\}$. This is easily seen because if $a(t) = \int_t^\infty d\mu(s)$ (assuming $\lim_{t \rightarrow \infty} a(t) = 0$) where μ is a nonnegative bounded measure, then

$$\hat{a}(iy) = \frac{1}{y} \int_0^\infty \sin(yt) d\mu(t) - i \frac{1}{y} \int_0^\infty (1 - \cos(yt)) d\mu(t), \quad y \neq 0. \quad (6)$$

Then it follows directly from (1) that

$$\hat{r}(s) = \frac{\hat{a}(s)}{1 + \hat{a}(s)}, \quad \operatorname{Re} s \geq 0, s \neq 0. \quad (7)$$

Invoking the closed-graph theorem we see that it is sufficient to show that the mapping: $f \rightarrow \int_0^t r(t-s)f(s) ds$ is not necessarily continuous from $L^2(R_+)$ to $L^2(R_+)$. Consequently (7) implies that the proof will be completed if we find a function a such that (4) holds and

$$\limsup_{y \rightarrow 0} \left| \frac{a(iy)}{1 + a(iy)} \right| = +\infty. \quad (8)$$

To see this, assume that the above mentioned mapping is continuous from $L^2(R_+)$ to $L^2(R_+)$ and put $h(t) = \int_0^t r(t-s)f(s) ds$. Then we would clearly have

$$\hat{h}(iy) = \hat{a}(iy)(1 + \hat{a}(iy))^{-1} \hat{f}(iy)$$

where the Fourier transforms of h and f are taken in the L^2 -sense. By (8) and the continuity of $\hat{a}(iy)(1 + \hat{a}(iy))^{-1}$, $y \neq 0$, there exist for any n numbers ε_n and y_n , $0 < \varepsilon_n < y_n$, such that $|\hat{a}(iy)(1 + \hat{a}(iy))^{-1}| \geq n$ when $|y - y_n| \leq \varepsilon_n/2$. Fix n and take f such that $|\hat{f}(iy)| \geq \varepsilon_n^{-1/2}$ when $|y - y_n| \leq \varepsilon_n/2$, but $\|f\|_{L^2(R_+)}$ is bounded independently of n (use the Hilbert transform). Now we conclude that $\|\hat{h}\|_{L^2(R)} \geq n$ which gives a contradiction by the Plancherel theorem.

Now we are going to construct a nonnegative, bounded measure μ and a sequence $\{y_m\}_{m=1}^\infty$ such that if

$$a(t) = \int_t^\infty d\mu(s), \quad t \in R_+, \quad (9)$$

then

$$\operatorname{Re} \hat{a}(iy_m) = -1, \quad m \geq 2, \quad \operatorname{Im} \hat{a}(iy_m) \rightarrow 0 \quad \text{as } m \rightarrow \infty. \quad (10)$$

First we define a sequence of integers

$$q_{m+1} = 3q_m + m, \quad m \geq 0, \quad q_0 = 4. \quad (11)$$

The sequence $\{y_m\}_{m=1}^{\infty}$ will be defined by

$$y_m = 2^{-q_m}, \quad m \geq 1. \quad (12)$$

The measure μ will be (δ is the measure defined by $\delta(E) = 1$ if $0 \in E$ and $\delta(E) = 0$ otherwise)

$$\mu(E) = \sum_{n=1}^{\infty} c_n \delta(E - a_n) \quad (13)$$

where

$$a_m = 2\pi(2^{q_m} - 2^{q_m - 2q_{m-1}}), \quad m \geq 1, \quad (14)$$

and

$$c_m = (\sin b_m)^{-1} \left(y_m + \sum_{n=1}^{m-1} c_n \sin(y_m a_n) \right), \quad m \geq 2, \quad c_1 = 1 \quad (15)$$

where

$$b_m = 2\pi \cdot 2^{-2q_{m-1}}, \quad m \geq 1. \quad (16)$$

First we must check that μ , as defined above, really is a nonnegative bounded measure. It is easily seen that all the numbers c_m are positive. Obviously there exist constants k_1 and k_2 such that

$$\begin{aligned} c_m &\leq k_1 \left(\max_{1 \leq n \leq m-1} c_n \right) y_m b_m^{-1} \sum_{n=1}^{m-1} a_n \\ &\leq k_2 \left(\max_{1 \leq n \leq m-1} c_n \right) y_m b_m^{-1} a_{m-1} \end{aligned} \quad (17)$$

since the numbers a_m are increasing and $(m-2)a_{m-2}a_{m-1}^{-1}$ remains bounded by (11) and (14). As we clearly have $y_m a_{m-1} b_m^{-1} \leq 2^{-q_m + q_{m-1} + 2q_{m-1}} = 2^{1-m}$ by (11), (12), (14) and (16) it follows easily from (17) that

$$c_m = O(2^{-m}) \quad \text{as } m \rightarrow \infty. \quad (18)$$

This fact shows that the measure μ is bounded and so the function a is well defined by (9) and satisfies (4).

Now we proceed to study the numbers $\hat{a}(iy_m)$. Note that $y_m a_n \equiv 0 \pmod{2\pi}$ if $n > m$ and that $y_m a_m = 2\pi - b_m$. Hence it follows from (6) that

$$\operatorname{Re} \hat{a}(iy_m) = y_m^{-1} \left(\sum_{n=1}^{m-1} c_n \sin(y_m a_n) + c_m \sin(2\pi - b_m) \right) \quad (19)$$

and invoking (15) in (19) we conclude that

$$\operatorname{Re} \hat{a}(iy_m) = -1, \quad m \geq 2. \quad (20)$$

Utilizing the same properties of the sequences $\{y_m\}_{m=1}^{\infty}$ and $\{a_m\}_{m=1}^{\infty}$ as above it follows from (6) that

$$\begin{aligned} -\operatorname{Im}(\hat{a}y_m) &= y_m^{-1} \sum_{n=1}^{m-1} c_n (1 - \cos(y_m a_n)) \\ &\quad + c_m y_m^{-1} (1 - \cos(2\pi - b_m)). \end{aligned} \quad (21)$$

As $1 - \cos x = O(x^2)$ and since $\sum_{n=1}^{\infty} |c_n| < \infty$ we obtain by (11), (12), (14), (16), (18) and (21)

$$\begin{aligned} \operatorname{Im} \hat{a}(iy_m) &= O(y_m a_{m-1}^2) + O(c_m b_m^2 y_m^{-1}) \\ &= O(2^{-q_m + 2q_{m-1}}) + O(2^{-m - 4q_{m-1} + q_m}) \\ &= O(2^{-q_{m-1}}). \end{aligned} \quad (22)$$

Now (10) follows from (20) and (22) and as (8) follows from (10) the proof of the theorem is completed.

NOTE ADDED IN PROOF. A related result is contained in I. A. Ibragimov, *A remark on the ergodic theorem for Markov chains*, Theor. Probability Appl. **20** (1975), 174–176.

REFERENCES

1. G. Gripenberg, *Integrability properties of resolvents of Volterra equations*, Report-HTKK-MAT-A117, Helsinki Univ. of Technology, 1978.
2. S. I. Grossman, *Integrability of resolvents of certain Volterra integral equations*, J. Math. Anal. Appl. **48** (1974), 785–793.
3. G. S. Jordan and R. L. Wheeler, *A generalization of the Wiener-Lévy theorem applicable to some Volterra equations*, Proc. Amer. Math. Soc. **57** (1976), 109–114.
4. J. J. Levin, *Resolvents and bounds for linear and nonlinear Volterra equations*, Trans. Amer. Math. Soc. **228** (1977), 207–222.
5. R. K. Miller, *On Volterra integral equations with nonnegative integrable resolvents*, J. Math. Anal. Appl. **22** (1968), 319–340.
6. R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain*, Amer. Math. Soc. Colloq. Publ., vol. 19, Amer. Math. Soc., Providence, R. I., 1934; reprinted 1973.
7. D. F. Shea and S. Wainger, *Variants of the Wiener-Lévy theorem, with applications to stability problems for some Volterra integral equations*, Amer. J. Math. **97** (1975), 312–343.
8. O. J. Staffans, *On the integrability of the resolvent of a Volterra equation*, Report-HTKK-MAT-A103, Helsinki Univ. of Technology, 1977.

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