THAT QUASINILPOTENT OPERATORS ARE NORM-LIMITS OF NILPOTENT OPERATORS REVISITED

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ABSTRACT. A new short proof is given that every quasinilpotent operator on a separable, infinite dimensional, complex Hilbert space is a norm-limit of nilpotent operators.

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$.

In [2] it was shown that every quasinilpotent operator T in $\mathcal{L}(\mathcal{H})$ (i.e., every T whose spectrum is the singleton $\{0\}$) is a norm-limit of nilpotent operators. The purpose of this note is to give a very short and easily remembered proof of this theorem. Our proof has two basic ingredients: the use of a model for quasinilpotent operators constructed in [4], and the use of a deep theorem of Voiculescu [7]. For completeness we review the relevant facts from [4] and [7] that we shall need.

Let $\mathcal{H}_\infty = \mathcal{H} \oplus \mathcal{H} \oplus \dots$ be the direct sum of S_0 copies of $\mathcal{H}$ indexed by the positive integers, let $\kappa = \{\kappa_n\}_{n=1}^\infty$ be a monotone decreasing sequence of nonnegative numbers converging to zero, and let U_κ denote the quasinilpotent backward weighted shift operator in $\mathcal{L}(\mathcal{H}_\infty)$ defined by the equation

$$U_\kappa(x_1, x_2, \dots, x_n, \dots) = (\kappa_1 x_2, \kappa_2 x_3, \dots, \kappa_n x_{n+1}, \dots). \quad (1)$$

THEOREM A [4]. *If T is any quasinilpotent operator in $\mathcal{L}(\mathcal{H})$, then there is a sequence $\kappa = \kappa(T) = \{\kappa_n\}_{n=1}^\infty$ with the above described properties and a subspace $\mathcal{M}$ of $\mathcal{H}_\infty$ that is invariant under the quasinilpotent operator U_κ such that T is similar to $U_\kappa|_{\mathcal{M}}$ and such that the subspace $\mathcal{H}_\infty \ominus \mathcal{M}$ is infinite dimensional.*

We shall need some lemmas, the first two of which are completely elementary.

LEMMA 1. *Let T_1 and T_2 belong to $\mathcal{L}(\mathcal{H}_1)$ and $\mathcal{L}(\mathcal{H}_2)$, respectively, and suppose that there exists a sequence $\{S_n\}_{n=1}^\infty$ of invertible operators $S_n: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that $\|S_n^{-1}T_1S_n - T_2\| \rightarrow 0$. If T_1 is a norm-limit of nilpotent operators, then so is T_2 .*

LEMMA 2. *The operator U_κ in (1) is a norm-limit of nilpotent operators in $\mathcal{L}(\mathcal{H}_\infty)$ (since the sequence κ converges to zero).*

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The following lemma is only slightly less elementary and is a special case of [6, Proposition 6.18].

LEMMA 3. Let $S = (S_{ij})_{i,j=1}^n$ be an $n \times n$ matrix with operator entries acting on the direct sum $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ of n copies of $\mathcal{H}$ and having the property that $S_{ij} = 0$ if $i > j$. If each S_{ii} , $1 \leq i \leq n$, is a norm-limit of nilpotent operators, then so is S . On the other hand, if S is a norm-limit of nilpotent operators, then so is $S_{11} \oplus \cdots \oplus S_{nn}$.

PROOF. For $1 \leq i \leq n$, let N_{ii} be a nilpotent operator such that $\|N_{ii} - S_{ii}\| < \varepsilon$. Then the matrix $\tilde{S}$ obtained by replacing each diagonal entry S_{ii} by the corresponding operator N_{ii} satisfies $\|\tilde{S} - S\| < \varepsilon$, and it is easy to see that $\tilde{S}$ is nilpotent. To go the other way, observe that by choosing n sufficiently large, the matrix

$$\text{Diag}(1, n, \dots, n^{n-1})(S_{ij})\text{Diag}\left(1, \frac{1}{n}, \dots, \frac{1}{n^{n-1}}\right)$$

can be made as close as desired to $S_{11} \oplus \cdots \oplus S_{nn}$, and use Lemma 1.

LEMMA 4. If T belongs to $\mathcal{L}(\mathcal{H})$ and is quasinilpotent, and Theorem A is used to write the space $\mathcal{H}_\infty$ as $\mathcal{H}_\infty = \mathfrak{M} \oplus \mathfrak{M}^\perp$ where $U_\kappa \mathfrak{M} \subset \mathfrak{M}$ and T is similar to $T' = U_\kappa|_{\mathfrak{M}}$, then in the corresponding matrix decomposition

$$U_\kappa = \begin{pmatrix} T' & A \\ 0 & S \end{pmatrix}, \quad (2)$$

the operator S is quasinilpotent and $T' \oplus S$ is a norm-limit of nilpotent operators.

PROOF. That S is quasinilpotent follows from the fact that U_κ is quasinilpotent (Theorem A) and the form of the matrix (2). That $T' \oplus S$ is a norm-limit of nilpotent operators is an immediate consequence of Lemmas 2 and 3.

THEOREM B [1]. If T is any quasinilpotent operator in $\mathcal{L}(\mathcal{H})$, then $T \oplus 0$ is a norm-limit of nilpotent operators in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$.

PROOF. According to Lemmas 4 and 1, $T \oplus S$ is a norm-limit of nilpotent operators on $\mathcal{H} \oplus \mathcal{H}$ and S is quasinilpotent. Thus, by a theorem of Rota (cf. [5, p. 77]), S is similar to operators of arbitrarily small norm, and the result follows by another application of Lemma 1.

COROLLARY 1. Suppose T_1 and T_2 are quasinilpotent operators in $\mathcal{L}(\mathcal{H})$ and both T_1 and T_2^* have infinite dimensional kernels. Then $T_1 \oplus T_2$ is the norm-limit of nilpotent operators in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$.

PROOF. Up to unitary equivalence, $T_1 \oplus T_2$ may be written as a 4×4 matrix

$$\left(\begin{array}{cc|cc} 0 & B_1 & 0 & 0 \\ 0 & C_1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & B_2 & C_2 \end{array} \right).$$

acting on the Hilbert space $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$, and this matrix, in turn, is unitarily equivalent to

$$\left(\begin{array}{cc|cc} C_2 & 0 & 0 & B_2 \\ 0 & 0 & B_1 & 0 \\ \hline 0 & 0 & C_1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

Since C_1 and C_2 are quasinilpotent, the corollary now follows from Theorem B and Lemma 3.

We turn now to some preparatory notation that we shall need to review the above-mentioned theorem of Voiculescu. If T_1 and T_2 are operators in $\mathcal{L}(\mathcal{H}_1)$ and $\mathcal{L}(\mathcal{H}_2)$, respectively, and if there exist a unitary operator $U: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ and a compact operator K in $\mathcal{L}(\mathcal{H}_2)$ such that $UT_1U^* = T_2 + K$ and such that $\|K\| < \varepsilon$, then we say that T_1 and T_2 are ε -compalant and we write $T_1 \sim T_2(\varepsilon)$. Observe that the relation of ε -compalant is symmetric, that is, $T_1 \sim T_2(\varepsilon)$ if and only if $T_2 \sim T_1(\varepsilon)$. We denote the ideal of compact operators in $\mathcal{L}(\mathcal{H})$ by $\mathbf{K}(\mathcal{H})$, and the quotient map of $\mathcal{L}(\mathcal{H})$ onto the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathbf{K}(\mathcal{H})$ by π . If $T \in \mathcal{L}(\mathcal{H})$, we denote by $C_\varepsilon^*(T)$ the separable C^* -subalgebra of $\mathcal{L}(\mathcal{H})/\mathbf{K}(\mathcal{H})$ generated by $\pi(T)$ and $\pi(1_{\mathcal{H}})$.

THEOREM C [7]. *Let $T \in \mathcal{L}(\mathcal{H})$, and let ρ be a faithful representation of $C_\varepsilon^*(T)$ on a separable Hilbert space $\mathcal{K}$ such that $\rho(\pi(1_{\mathcal{H}})) = 1_{\mathcal{K}}$. Then for every positive number ε , $T \oplus \rho(\pi(T)) \sim T(\varepsilon)$.*

Based on the above considerations we are now prepared to give a transparent proof of the following theorem.

THEOREM D [2]. *Every quasinilpotent operator T in $\mathcal{L}(\mathcal{H})$ is a norm-limit of nilpotent operators.*

PROOF. Using Theorem C and Lemma 1, we observe that it suffices to construct some faithful representation ρ of $C_\varepsilon^*(T)$ on a separable Hilbert space $\mathcal{K}$ such that $\rho(\pi(1_{\mathcal{H}})) = 1_{\mathcal{K}}$ and such that $T \oplus \rho(\pi(T))$ is a norm-limit of nilpotent operators in $\mathcal{L}(\mathcal{H} \oplus \mathcal{K})$. Furthermore, if τ is a faithful representation of $C_\varepsilon^*(T)$ on a separable Hilbert space $\mathcal{U}$ such that $\tau(\pi(1_{\mathcal{H}})) = 1_{\mathcal{U}}$, then $\mathcal{K}$ may be taken to be $\mathcal{U} \oplus \mathcal{U}$ and ρ to be $\tau \oplus \tau$. Thus it suffices to find a representation τ with the above-mentioned properties such that $[T \oplus \tau(\pi(T))] \oplus \tau(\pi(T))$ is a norm-limit of nilpotent operators. On the other hand, by virtue of Corollary 1, it suffices to construct such a representation τ with the property that both $\tau(\pi(T))$ and $\tau(\pi(T^*))$ have infinite dimensional kernels, and this goes as follows. Since T and T^* are quasinilpotent, one

knows from [8] (cf. also [3, Theorem 3.1]) that there exist projections E and F in $\mathcal{L}(\mathcal{H})$ of infinite rank such that $\pi(T)\pi(E) = 0$ and $\pi(T^*)\pi(F) = 0$. Furthermore, E may be written as $E = \sum_{n=1}^{\infty} E_n$, where $\{E_n\}$ is an orthogonal sequence of projections each of which has infinite rank, and similarly F may be written as $F = \sum_{n=1}^{\infty} F_n$. Let $\mathcal{Q}$ be the (separable) C^* -subalgebra of $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ generated by $\pi(T)$, $\pi(1_{\mathcal{H}})$, and the countable families $\{\pi(E_n)\}_{n=1}^{\infty}$ and $\{\pi(F_n)\}_{n=1}^{\infty}$, and let ν be a faithful representation of $\mathcal{Q}$ on a separable Hilbert space $\mathcal{U}$ so chosen that $\nu(\pi(1_{\mathcal{H}})) = 1_{\mathcal{U}}$. Then $\nu(\pi(T))\nu(\pi(E_n)) = 0$ and $\nu(\pi(T^*))\nu(\pi(F_n)) = 0$ for all positive integers n , so clearly the operators $\nu(\pi(T))$ and $\nu(\pi(T^*)) = (\nu(\pi(T)))^*$ have infinite dimensional kernels, and the result follows by setting $\tau = \nu|_{C_e^*(T)}$.

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