DERIVATIONS AND THE TRACE-CLASS OPERATORS

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ABSTRACT. Let $\mathcal{R}(\Delta_A)$ represent the range of the derivation generated by $A \in \mathcal{B}(\mathcal{H})$. It is shown that for each $n > 2$, $\text{tr}(T^n) = 0$ for any trace-class operator $T \in \{A\}'$ which is either (a) the weak limit of a sequence in $\mathcal{R}(\Delta_A)$ or (b) a finite rank operator in the weak closure of $\mathcal{R}(\Delta_A)$. From this it follows that if $K \in \{A\}'$ is a compact operator in the weak closure of $\mathcal{R}(\Delta_A)$, then K is quasinilpotent.

1. Introduction. Let $\mathcal{H}$ be a separable Hilbert space and Δ_A the derivation generated by $A \in \mathcal{B}(\mathcal{H})$ (the bounded linear operator on $\mathcal{B}(\mathcal{H})$ defined by $\Delta_A(X) = AX - XA$ for all $X \in \mathcal{B}(\mathcal{H})$). Kleinecke [6] has shown that $\{A\}' \cap \mathcal{R}(\Delta_A)$ consists only of quasinilpotents and H. W. Kim [5] extended this to compact operators which are in both $\{A\}'$ and the weak sequential closure of $\mathcal{R}(\Delta_A)$. We will extend this result to show that compact operators in both $\{A\}'$ and the weak closure of $\mathcal{R}(\Delta_A)$ are quasinilpotent. Also necessary and sufficient conditions are given in order for the weak closure of the range of one derivation to be contained in another.

2. Notation. Let $\mathcal{T}$ be the ideal of trace-class operators, that is, all compact operators $T \in \mathcal{B}(\mathcal{H})$ for which the eigenvalues of $(T^*T)^{1/2}$ (counted according to multiplicity) are summable. The ideal $\mathcal{T}$ admits a complex valued function $\text{tr}(T)$ which has the characteristic properties of the trace for matrices. The trace function is defined by $\text{tr}(T) = \sum_n (\langle Te_n, e_n \rangle)$ where $\{e_n\}$ is any complete orthonormal sequence in $\mathcal{H}$. As Banach spaces, $\mathcal{T}$ may be identified with the conjugate space of the ideal $\mathcal{K}$ of compact operators by means of the linear isometry $T \rightarrow f_T$ where $f_T(X) = \text{tr}(XT)$. Moreover, $\mathcal{B}(\mathcal{H})$ is the conjugate space of $\mathcal{T}$. The weak* (or ultraweakly) continuous linear functionals on $\mathcal{B}(\mathcal{H})$ are those of the form f_T for some $T \in \mathcal{T}$. Furthermore, the weak (or weak operator topology) continuous linear functionals on $\mathcal{B}(\mathcal{H})$ are those of the form f_T where T is of finite rank. (For a discussion of the above remarks see Dixmier [1].)

In the following, $\mathcal{R}(\Delta_A)^-$ will denote the weak closure and $\mathcal{R}(\Delta_A)^{\omega}$ will denote the weak* closure of the range of derivation generated by $A \in \mathcal{B}(\mathcal{H})$.

3. We begin with the following result on the trace of certain operators.

THEOREM 1. *Let $A \in \mathcal{B}(\mathcal{H})$ and let T be a trace-class operator which*

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commutes with A . If either

(a) T is the weak limit of a sequence of elements in $\mathfrak{R}(\Delta_A)$ or

(b) T is of finite rank and in the weak closure of $\mathfrak{R}(\Delta_A)$

then $\text{tr}(T^n) = 0$ for all $n \geq 2$.

PROOF. If the trace-class operator T commutes with A , then

$$\begin{aligned} f_T(AX - XA) &= \text{tr}(T(AX - XA)) = \text{tr}(TAX - TXA) \\ &= \text{tr}(TAX - TAX) = 0. \end{aligned}$$

Hence f_T is a weak* continuous linear functional which vanishes on $\mathfrak{R}(\Delta_A)$. Now if $\{AX_n - X_nA\}$ converges weakly to T then $\|AX_n - X_nA\|$ is bounded (see [2, Problem 40]) and since the weak and weak* topologies are equivalent on bounded sets,

$$f_T(T) = \text{tr}(T^2) = 0.$$

If T is of finite rank and commutes A , then f_T is a weakly continuous linear function which vanishes on $\mathfrak{R}(\Delta_A)$. Hence if T is also in the weak closure of $\mathfrak{R}(\Delta_A)$ then

$$f_T(T) = \text{tr}(T^2) = 0.$$

In either case, since $T^n \mathfrak{R}(\Delta_A) \subset \mathfrak{R}(\Delta_A)$,

$$\text{tr}(T^n) = \text{tr}(TT^{n-1}) = f_T(T^{n-1}) = 0$$

for all $n \geq 2$.

The set of Hilbert-Schmidt operators ($\mathfrak{S}$) on $\mathfrak{H}$ equipped with the trace inner product $(A, B) = \text{tr}(AB^*)$ is a Hilbert space which contains $\mathfrak{T}$ [7, p. 34]. In the following theorem $\mathfrak{R}(\Delta_A)^\sim$ will denote the Hilbert-Schmidt closure of the range of a derivation generated by a Hilbert-Schmidt operator A .

THEOREM 2. If $A \in \mathfrak{S}$ and $T \in \{A\}' \cap \mathfrak{R}(\Delta_A)^\sim$ then $\text{tr}(T^n) = 0$ for all $n \geq 2$.

PROOF. The restriction of Δ_A to $\mathfrak{S}$ is a bounded linear operator on $\mathfrak{S}$ with adjoint equal to Δ_{A^*} restricted to $\mathfrak{S}$. Hence $\mathfrak{S} = \mathfrak{R}(\Delta_A|_{\mathfrak{S}})^\sim \oplus \{A^*\}' \cap \mathfrak{S}$. Also, $\mathfrak{R}(\Delta_A|_{\mathfrak{S}})^\sim = \mathfrak{R}(\Delta_A)^\sim$ (see Theorem 2 of [8]). Therefore,

$$\text{tr}(T^2) = (T, T^*) = 0.$$

For $n > 2$, $T^{n-1} \in \{A\}' \cap \mathfrak{R}(\Delta_A)^\sim$. Hence

$$\text{tr}(T^n) = (T^{n-1}, T^*) = 0.$$

Operators satisfying the above theorems have a distinct spectral property which follows from the following lemma (the proof of which was suggested by Y. Ho).

LEMMA. If T is a trace-class operator with $\text{tr}(T^n) = 0$ for all $n \geq 1$ then $\sigma(T) = 0$.

PROOF. Assume $\sigma(T) \neq \{0\}$ and let $\lambda \neq 0$ be an isolated eigenvalue of

finite multiplicity. Also, let γ be a Jordan curve such that $\sigma(T)$ intersects the interior of γ only at λ . Then

$$E = \frac{1}{2\pi i} \int_{\gamma} (z - T)^{-1} dz$$

is the Riesz projection determined by γ . Hence $\lambda \in \sigma(ET)$ and $E \in \{T\}''$. By Runge's Theorem, there exists a sequence of polynomials $\{P_m\}$ such that

$$\lim \|P_m(T) - E\| = 0.$$

If $\text{tr}(T^n) = 0$ for all $n \geq 1$, then

$$\text{tr}((ET)^n) = \text{tr}(ET^n) = \lim_{m \rightarrow \infty} \text{tr}(P_m(T)T^n) = 0$$

for all $n \geq 1$. Since $E(\mathcal{H})$ is finite dimensional, Jacobson's Lemma (see [3, p. 273] for example) implies ET is nilpotent which contradicts the assumption that $\lambda \neq 0$.

COROLLARY. *If T satisfied either Theorem 1 or Theorem 2, then $\sigma(T) = 0$.*

PROOF. If T satisfies either of the theorems, then $\text{tr}(T^{2n}) = 0$ for all $n \geq 1$. It follows from the lemma that $\sigma(T^2) = 0$ and hence $\sigma(T) = 0$.

We are now ready to prove the following theorem.

THEOREM 3. *Let $A \in \mathfrak{B}(\mathcal{H})$. If K is a compact operator in $\{A\}' \cap \mathfrak{R}(\Delta_A)^-$, then K is quasinilpotent.*

PROOF. Assume $\lambda \in \sigma(K)$ with $\lambda \neq 0$ an isolated eigenvalue of finite multiplicity. Let E be the Riesz projection as defined in the proof of the lemma. Since $E \in \{K\}''$, then $EK \in \{A\}'$ and $E\mathfrak{R}(\Delta_A) \subset \mathfrak{R}(\Delta_A)$. Multiplication is weakly continuous with respect to one argument [2, Problem 92]. Hence $EK \in \mathfrak{R}(\Delta_A)^-$, and since EK has finite rank, it satisfies part (b) of Theorem 1. By the corollary, $\sigma(EK) = 0$ which contradicts the assumption that $\lambda \neq 0$.

4. The above leads us to consider the relationship between the closures of two derivation ranges. In the following theorem we let $\mathfrak{F}$ denote the ideal of finite rank operators.

THEOREM 4. *Let $A, B \in \mathfrak{B}(\mathcal{H})$. Then*

- (a) $\mathfrak{R}(\Delta_B)^- \subset \mathfrak{R}(\Delta_A)^-$ if and only if $\{A\}' \cap \mathfrak{F} \subset \{B\}' \cap \mathfrak{F}$ and
- (b) $\mathfrak{R}(\Delta_B)^{\circ\circ} \subset \mathfrak{R}(\Delta_A)^{\circ\circ}$ if and only if $\{A\}' \cap \mathfrak{F} \subset \{B\}' \cap \mathfrak{F}$.

PROOF. To prove (a) assume $\{A\}' \cap \mathfrak{F} \subset \{B\}' \cap \mathfrak{F}$ and let f_T be a weakly continuous linear functional which annihilates $\mathfrak{R}(\Delta_A)$, then

$$0 = f_T(AX - XA) = \text{tr}(T(AX - XA)) = \text{tr}((TA - AT)X)$$

for all $X \in \mathfrak{B}(\mathcal{H})$, hence $TA = AT$ and $T \in \{A\}' \cap \mathfrak{F} \subset \{B\}' \cap \mathfrak{F}$. Therefore,

$$f_T(BX - XB) = \text{tr}(T(BX - XB)) = 0.$$

Hence f_T annihilates $\mathfrak{R}(\Delta_B)$ and we have $\mathfrak{R}(\Delta_B)^- \subset \mathfrak{R}(\Delta_A)^-$.

For the opposite implication we reverse the above argument. To prove part (b) we replace $T \in \mathfrak{F}$ by $T \in \mathfrak{T}$.

A related question is if proper containment ever occurs between the weak* and weak closures of the range of a derivation. The following provides us with an example.

For $A \in \mathfrak{B}(\mathfrak{H})$, J. P. Williams [9] has shown that $\mathfrak{R}(\Delta_A)^- = \mathfrak{B}(\mathfrak{H})$ if and only if $\{A\}' \cap \mathfrak{F} = \{0\}$ and $\mathfrak{R}(\Delta_A)^- = \mathfrak{B}(\mathfrak{H})$ if and only if $\{A\}' \cap \mathfrak{T} = \{0\}$.

EXAMPLE. Let V be the Volterra operator on $L^2[0, 1]$ defined by $(Vf)(x) = \int_0^x f(t) dt$. The invariant subspaces of V are the subspaces $L^2[a, 1]$ for $0 \leq a < 1$ (see [4]). Therefore $\{V\}' \cap \mathfrak{F} = \{0\}$ and $\mathfrak{R}(\Delta_V)^- = \mathfrak{B}(\mathfrak{H})$. Furthermore, $\{V\}'$ does contain operators in $\mathfrak{T}$ (V^2 for example), hence $\mathfrak{R}(\Delta_V)^- \neq \mathfrak{B}(\mathfrak{H})$.

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