A NOTE ON AN INVERSE SCATTERING PROBLEM¹

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ABSTRACT. Here it is shown that the inverse scattering problem of a plane wave incident off a spherically symmetric medium is an improperly posed problem and that if a continuous index of refraction exists, then it is unique.

1. Introduction. Recently the scattering of a plane wave by an inhomogeneous medium has received considerable attention (see [1], [2] and [4] and the references given there). Using a result of Kleinman [3] for obtaining from the scattered field a function which is regular at infinity, Rorres [4] gives a practical method for constructing the solution, for small values of the wave number. Colton [2] gives a very interesting function theoretic method for obtaining the solution for all values of the wave number, provided one is able to calculate a Riemann function to a certain hyperbolic partial differential equation. Ahner [1] represents the solution as a uniformly convergent Neumann series for small values of the wave number, and shows that his method compares favorably with the method of Rorres. Colton assumes that his index of refraction is twice continuously differentiable in E^3 , whereas Rorres and Ahner assume merely that this function is continuous in E^3 .

For the case of a spherically symmetric inhomogeneous medium, Rorres derives an important connection between the scattering amplitude function and various moments of the index of refraction. This result was also obtained by Ahner and by Colton using their respective methods. Using this result, an analysis is given in this paper of the inverse scattering problem of a plane wave incident off a spherically symmetric inhomogeneous medium. Hereafter, we shall denote this problem by RISP (radially symmetric inverse scattering problem). It is rigorously shown here that this problem is improperly posed and that if a continuous index of refraction exists, then it is unique. The important question of existence will be analyzed in a later paper.

2. Background. Consider the problem (P) of a plane wave incident off an inhomogeneous medium of finite extent:

$$\begin{aligned} [\Delta + k^2 n(\mathbf{x})]u(\mathbf{x}) &= 0, & u(\mathbf{x}) &= e^{ik \cdot \mathbf{x}} + u^s(\mathbf{x}) \\ \lim_{r \rightarrow \infty} r \left(\frac{\partial}{\partial r} - ik \right) u^s &= 0, & u(\mathbf{x}) &\in C^2(E^3), n(\mathbf{x}) \equiv 1 \text{ for } r \geq 1 \end{aligned}$$

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where $n(\mathbf{x})$ is a real continuous function in E^3 . For the direct scattering problem, $n(\mathbf{x})$ is given and one wishes to find $u(\mathbf{x})$. For the inverse scattering problem, the scattering amplitude function is given for some specified values of the wave number k , say for $0 < k < c$, where c is a fixed constant, and one wishes to find, if at all possible, $n(\mathbf{x})$ for $r = |\mathbf{x}| \leq 1$ in some specified function space. We take $C[0, 1]$ as our space.

Let (r, θ, ϕ) represent the spherical coordinates of the point $\mathbf{x} \in E^3$. For the case of a spherically symmetric inhomogeneous medium, i.e., $n(\mathbf{x}) = n(r)$, the scattering amplitude function, defined by

$$S(\theta, k) = \lim_{r \rightarrow \infty} r e^{-ikr} u^s(r, \theta) \quad (2.1)$$

is continuously differentiable and has the representation

$$S(\theta, k) = \sum_{n=0}^{\infty} A_n(k) P_n(\cos \theta). \quad (2.2)$$

It can be shown (e.g., see [1, Theorem 4.3]) that

$$A_n(k) = \left\{ -\frac{1}{2n+1} \left(\frac{2^n n!}{(2n)!} \right)^2 \int_0^1 m(r) r^{2n+2} dr \right\} k^{2n+2} + O(k^{2n+4}) \quad (2.3)$$

where $m(r) = 1 - n(r)$. (We point out that for convenience, we have taken $a = 1$ here, where a is the radius of the inhomogeneity in [1].) Hereafter, we shall refer to $m(r)$ as the density function.

3. Main results. In this section it is shown that RISP is an improperly posed problem and that if a continuous index of refraction exists, then it is unique. First we recall Müntz's Theorem (see [5, p. 305]). The infinite sequence of powers,

$$1, r^{\lambda_1}, r^{\lambda_2}, \dots$$

with positive exponents which approach infinity, has a dense span in $C[0, 1]$ if and only if $\sum_{j=1}^{\infty} 1/\lambda_j$ diverges.

We now establish the following important result:

LEMMA. Let $m(r)$ and $\tilde{m}(r)$ be continuous density functions corresponding to spherically symmetric inhomogeneous mediums and let

$$S(\theta, k) = \sum_{n=0}^{\infty} A_n(k) P_n(\cos \theta) \quad \text{and} \quad \tilde{S}(\theta, k) = \sum_{n=0}^{\infty} \tilde{A}_n(k) P_n(\cos \theta)$$

be their corresponding scattering amplitude functions. Let Λ be any set of nonnegative integers which satisfy

$$\Lambda = \left\{ n \mid n = 0, \lambda_1, \lambda_2, \dots, \text{ such that } \sum_{j=1}^{\infty} \frac{1}{\lambda_j} = \infty \right\}.$$

If $A_n(k) = \tilde{A}_n(k)$ for $n \in \Lambda$, then $A_n(k) = \tilde{A}_n(k)$ for $n = 0, 1, 2, \dots$ and $m(r) \equiv \tilde{m}(r)$.

PROOF. Let $M(r) = m(r)r^2$ and $\tilde{M}(r) = \tilde{m}(r)r^2$ and let

$$\psi(r) = M(r) - \tilde{M}(r). \quad (3.1)$$

From (2.2) and (2.3) we have

$$\int_0^1 M(r)r^{2n} dr = \lim_{k \rightarrow 0} \left\{ -\frac{2n+1}{k^{2n+2}} \left(\frac{(2n)!}{2^n n!} \right)^2 A_n(k) \right\} \quad (3.2)$$

$$= \lim_{k \rightarrow 0} \left\{ -\frac{(2n+1)^2}{2k^{2n+2}} \left(\frac{(2n)!}{2^n n!} \right)^2 \int_0^\pi S(\theta, k) P_n(\cos \theta) \sin \theta d\theta \right\}. \quad (3.3)$$

Similar results hold for $\tilde{M}(r)$. It follows that

$$0 = \int_0^1 \psi(r)r^{2n} dr, \quad n \in \Lambda. \quad (3.4)$$

By Müntz's Theorem, the span of $1, r^{2\lambda_1}, r^{2\lambda_2}, \dots$ is dense in $C[0, 1]$; hence for any $\varepsilon > 0$, there is a polynomial $Q(r) = \sum_{j=0}^N a_j r^{2\lambda_j}$, where $\lambda_0 = 0$, such that

$$|\psi(r) - Q(r)| < \varepsilon. \quad (3.5)$$

From (3.4)

$$\int_0^1 \psi(r)Q(r) dr = \sum_{j=0}^N a_j \int_0^1 \psi(r)r^{2\lambda_j} dr = 0, \quad (3.6)$$

and it follows that

$$\int_0^1 \psi^2(r) dr = \int_0^1 \psi(r)[\psi(r) - Q(r)] dr < \varepsilon \int_0^1 |\psi(r)| dr \quad (3.7)$$

for any $\varepsilon > 0$. Thus $\psi(r) \equiv 0$ and it is seen that $\tilde{A}_n(k) = A_n(k)$ for all n and the proof of the lemma is complete.

From this lemma, we now establish

THEOREM 1. *RISP is not a properly posed problem. Specifically, there exists a continuously differentiable function $\tilde{S}(\theta, k)$, possessing a Legendre series expansion of the form (2.2), which cannot be generated by a continuous index of refraction.*

PROOF. Consider the scattering amplitude function $S(\theta, k)$, together with its coefficients $A_n(k)$, corresponding to a given continuous density function $m(r)$. Now form new coefficients $\tilde{A}_n(k)$ by altering the $A_n(k)$'s on a finite set Φ of integers n , with zero not in Φ , while taking $\tilde{A}_n(k) = A_n(k)$ when $n \notin \Phi$. Clearly the corresponding function

$$\tilde{S}(\theta, k) = \sum_{n=1}^{\infty} \tilde{A}_n(k) P_n(\cos \theta) \quad (3.8)$$

will still be continuously differentiable, since its Legendre series representation differs from that of the known continuously differentiable function

$S(\theta, k)$ by only finitely many terms. This function $\tilde{S}(\theta, k)$ cannot, however, be generated by a continuous density function $\tilde{m}(r)$; because if it were, an application of the above lemma, with Λ taken as the complement of the finite set of integers Φ , would enable us to conclude that $\tilde{A}_n(k) = A_n(k)$ for all n , contrary to the fact that $\tilde{A}_n(k) \neq A_n(k)$ when $n \in \Phi$. This completes the proof of the theorem.

Our last result is

THEOREM 2. *If there exists a continuous density function to RISIP, then it is unique.*

PROOF. Suppose $m(r)$ and $\tilde{m}(r)$ are continuous densities having the same scattering amplitude function $S(\theta, k)$ with representation (2.2). Applying the lemma with $\Lambda = \{0, 1, 2, \dots\}$, it follows that $m(r) = \tilde{m}(r)$.

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