SLICING THE CUBE IN $\mathbb{R}^n$ AND PROBABILITY (BOUNDS FOR THE MEASURE OF A CENTRAL CUBE SLICE IN $\mathbb{R}^n$ BY PROBABILITY METHODS)

DOUGLAS HENSLEY

ABSTRACT. A cube of dimension n and side 1 is cut by a hyperplane of dimension $n - 1$ through its center. The usual $n - 1$ measure of the intersection is bounded between 1 and M , independent of n . The proof uses an inequality for sums of independent random variables.

1. Introduction. If a square or a cube of side 1 is cut by a line (resp. plane) through its center then the length (area) of the intersection is ≥ 1 . But even for the cube this is not obvious.

Motivated perhaps by these examples, Dr. Anton Good conjectured that if a cube of dimension n were sliced by an $n - 1$ dimensional hyperplane through its center the usual $n - 1$ measure of the set of intersection would always be ≥ 1 . We proved this and an analogous upper bound. A curious feature of the solution to this straightforward problem of geometry is that it is largely based on probabilistic considerations. Let us then restate the problem as a question in probability.

The original version is this:

(1) Let $C = [-\frac{1}{2}, \frac{1}{2}]^n \subseteq \mathbb{R}^n$. Let B be an $n - 1$ dimensional subspace of $\mathbb{R}^n$. Let m denote the usual measure on B . Then is $m(B \cap C) \geq 1$?

If some component of the vector normal to B is zero the problem reduces to one in a lower dimension. So suppose that no component of that normal vector v is zero. Then after reindexing and redirecting the axis we can write

(2) $v = [v_1, v_2, \dots, v_n]$ with $v_1 \geq v_2 \geq \dots \geq v_n = 1$. Then $B \cap C = \{(x_1, x_2, \dots, x_n): x_1 v_1 + x_2 v_2 + \dots + x_n v_n = 0 \text{ and } |x_k| \leq \frac{1}{2}, 1 \leq k \leq n\}$.

For $1 \leq k \leq n - 1$ let X_k be a random variable, independent of the others, having constant density v_k^{-1} on $[(-\frac{1}{2})v_k, (\frac{1}{2})v_k]$. The projection of $B \cap C$ onto $x_n = 0$ is $\{(x_1, x_2, \dots, x_{n-1}): |\sum_{k=1}^{n-1} x_k v_k| \leq \frac{1}{2}\}$. Its measure equals $\text{Prob}(|\sum_{k=1}^{n-1} X_k| \leq \frac{1}{2})$. Thus $m(B \cap C) = |v| \text{Prob}(|\sum X_k| \leq \frac{1}{2})$. This problem is therefore equivalent to

(3) With v, X_k as defined above, is $|v| \text{Prob}(|\sum X_k| \leq \frac{1}{2}) \geq 1$?

REMARK. In a discussion of this problem at the Institute for Advanced Study I opined that taking $v = [1, 1, \dots, 1]$ and n large gave a counter-example. Professor Selberg put the problem in its probabilistic form and used

Received by the editors November 18, 1977 and, in revised form, April 27, 1978.

AMS (MOS) subject classifications (1970). Primary 52A20, 52A40, 60E05, 60D05.

Key words and phrases. Thickness of a random variable, hyperplane slicing a cube.

0002-9939/79/0000-0020/\$02.50

© 1979 American Mathematical Society

the central limit theorem to show that for my purported counterexample, $m(B \cap C) \sim (6/\pi)^{1/2}$ as $n \rightarrow \infty$. This was a surprise to both of us as neither had expected a value close to 1.

2. An inequality in probability. Suppose f is a probability density function of a random variable X . If f is nondecreasing on $[-\infty, 0]$ and symmetric about 0 we say X is a *good* random variable.

DEFINITION. If X and Y are independent and good random variables and if for all $a \geq 0$, $\text{Prob}(|X| \leq a) > \text{Prob}(|Y| \leq a)$ then X is *more peaked* than Y and we write $X < Y$.

THEOREM 1 (Z. W. BIRNBAUM [1]). Suppose X, Y, Z , and Z' are independent good random variables, $X < Y$, and Z and Z' are identically distributed. Then $X + Z$ and $Y + Z'$ are good, are independent, and $X + Z < Y + Z'$.

The immediate corollaries $\mathcal{Q}$ and $\mathfrak{B}$ below are the ones we shall use later.

COROLLARY $\mathcal{Q}$. If X_1, X_2, Y_1 and Y_2 are independent and good, $X_1 < Y_1$, $X_2 < Y_2$ then $X_1 + X_2 < Y_1 + Y_2$, the sums are independent and both sums are good.

COROLLARY $\mathfrak{B}$. If $X_1, X_2, \dots, X_{n-1}, Y_1, Y_2, \dots, Y_{n-1}$ are independent, all are good, if for $1 \leq k \leq n-1$, $X_k < Y_k$, and if $X_0 = \sum_{k=1}^{n-1} X_k$ and $Y_0 = \sum_{k=1}^{n-1} Y_k$, then X_0 and Y_0 are good and $X_0 < Y_0$.

Theorem 1 has since been generalized [4].¹

3. Application to cube slicing.

THEOREM 2. Suppose $v_1 \geq v_2 \geq \dots \geq v_n = 1$ and suppose $X_k, 1 \leq k \leq n-1$, are independent random variables uniformly distributed on $[-\frac{1}{2}v_k, \frac{1}{2}v_k]$. Let $\mathbf{v} = [v_1, v_2, \dots, v_n]$ and let $X_0 = \sum_{k=1}^{n-1} X_k$. Then

$$|\mathbf{v}| \text{Prob}(|X_0| \leq \tfrac{1}{2}) > 1. \quad (8)$$

PROOF. Let $Y_k, 1 \leq k \leq n-1$, be such that $\{X_k\} \cup \{Y_k\}$ are independent and each Y_k has distribution $\mathcal{N}(0, \sigma_k)$ where $\sigma_k = (2\pi)^{-1/2}v_k$. Let $Y_0 = \sum_{k=1}^{n-1} Y_k$. Then all X_k and Y_k are good and for each k , $X_k < Y_k$. Thus by Corollary $\mathfrak{B}$, $X_0 < Y_0$. Now Y_0 is $\mathcal{N}(0, \sigma_0)$ where $\sigma_0^2 = \sum_{k=1}^{n-1} \sigma_k^2$.

Thus $\text{Prob}(|Y_0| \leq \tfrac{1}{2})$ equals

$$(2\pi)^{-1/2} \sigma_0^{-1} \int_{-1/2}^{1/2} \exp(-\tfrac{1}{2}x^2/\sigma_0^2) dx. \quad (9)$$

Let $\mathbf{u} = [v_1, v_2, \dots, v_{n-1}]$. Then

$$|\mathbf{v}| \text{Prob}(|Y_0| \leq \tfrac{1}{2}) = (|\mathbf{v}|/|\mathbf{u}|) \int_{-1/2}^{1/2} \exp(-\pi x^2/|\mathbf{u}|^2) dx. \quad (10)$$

¹Also M. Kanter, *Unimodality and dominance for symmetric random vectors*, Trans. Amer. Math. Soc. **229** (1977), 65–85.

Since the case $n = 2$ is trivial we assume here that $n \geq 3$ so that $|\mathbf{u}|^2 \geq 2$. Let $1 + s = |\mathbf{v}|^2/|\mathbf{u}|^2$. Then $s = |\mathbf{u}|^{-2}$ and the expression in (10) equals

$$(1 + s)^{1/2} \int_{-1/2}^{1/2} \exp(-\pi s x^2) dx = H(s), \quad \text{say.} \quad (11)$$

Then $H(0) = 1$ so we can complete the proof by showing $dH/ds > 0$ for $0 < s < \frac{1}{2}$. We have

$$dH/ds = \frac{1}{2}(1 + s)^{-1/2} \int_{-1/2}^{1/2} (1 - 2(1 + s)\pi x^2) \exp(-\pi s x^2) dx. \quad (12)$$

By parts this is

$$\begin{aligned} & 2(x - (2/3)(1 + s)\pi x^3) \exp(-\pi s x^2) \Big|_0^{1/2} \\ & + 2 \int_0^{1/2} 2\pi s x (x - (2/3)(1 + s)\pi x^3) \exp(-\pi s x^2) dx. \end{aligned} \quad (13)$$

Both terms are greater than zero by inspection. $\square$

In terms of the original problem (1), Theorem 2 reads as follows.

THEOREM 2'. *The usual measure of $B \cap C$ is ≥ 1 and equality occurs only if B is parallel to some face of C .*

These results do not generalize in any way that is obvious to me to the more general:

CONJECTURE (A. Good). If C is $[-\frac{1}{2}, \frac{1}{2}]^n$ and B is a vector subspace of $\mathbb{R}^n$ with dimension $d < n$ then the usual measure m of $B \cap C$ is ≥ 1 .²

4. Upper bounds. A hyperplane not through the origin may miss the cube C altogether. Thus for nontrivial lower bounds we had to restrict ourselves to central slices. We now drop that restriction.

THEOREM 3. *If B' is a hyperplane, parallel to B through 0, then $m(B' \cap C) \leq m(B \cap C)$.*

PROOF. Let $\mathbf{v} = (v_1, v_2, \dots, v_n)$ be a vector normal to B . Without loss of generality we may take $v_n = 1$ and all $v_k \geq 0$ for $1 \leq k \leq n$. Now let X_k be independent random variables each uniformly distributed on $[-\frac{1}{2}v_k, \frac{1}{2}v_k]$. Then $\sum_{k=1}^{n-1} X_k$ is a good random variable, so for all a ,

$$\text{Prob} \left[\left| \sum_{k=1}^{n-1} X_k - a \right| \leq \frac{1}{2} \right] \leq \text{Prob} \left[\left| \sum_{k=1}^{n-1} X_k \right| \leq \frac{1}{2} \right].$$

On multiplying both sides of the above by $|\mathbf{v}|$ we have the desired inequality. $\square$

REMARK. Thus we need only find upper bounds for $m(B \cap C)$ where B is an $(n - 1)$ dimensional hyperplane through the origin.

²But J. Vaaler has recently proved this conjecture.

THEOREM 4. *There exists $M > 0$ such that for any dimension n , any B of dimension $n - 1$, $m(B \cap C) \leq M$. M may be taken to be 5.*

PROOF. Suppose $\mathbf{v} = [v_1, v_2, \dots, v_n]$ is a vector normal to B , and this time normalize, reindex, and redirect so that $v_1 = 1 \geq v_2 \geq \dots \geq v_n > 0$. Then with X_k , $2 \leq k \leq n$, as usual, $m(B \cap C) = |\mathbf{v}| \text{Prob}(|\sum_2^n X_k| < \frac{1}{2})$. Let $X = \sum_2^n X_k$.

Before proceeding let us agree to take $M \geq 5$. Now if $|\mathbf{v}|^2 \leq M$ then we are done as the probability in question is no greater than one. Thus we can assume $|\mathbf{v}|^2 > M^2$ and consequently $n > M^2$.

Now

$$\text{Prob}(|X| \leq \tfrac{1}{2}) = \int_{-1/2}^{1/2} f(s) ds \leq f(0),$$

where f is the density of X . Let L be the Fourier transform operator

$$Lg(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} g(x) e^{-ixt} dx.$$

Let $f_2, f_3, \dots, f_n$ be the density functions of $X_2, X_3, \dots, X_n$ respectively. Then for $2 \leq k \leq n$,

$$Lf_k(t) = (2\pi)^{-1/2} (\sin(\tfrac{1}{2} v_k t)) / (\tfrac{1}{2} v_k t). \quad (14)$$

Let “*” denote convolution, so that

$$g * h(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} g(x-y) h(y) dy.$$

Then the density of a sum of two independent random variables is $(2\pi)^{1/2}$ times the convolution of their densities. Thus by Fourier theory

$$f(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} Lf(t) e^{ixt} dt \quad (15)$$

and, in particular,

$$f(0) = (2\pi)^{-1} \int_{-\infty}^{\infty} \sum_{k=2}^n (\sin(\tfrac{1}{2} v_k t)) / (\tfrac{1}{2} v_k t) dt. \quad (16)$$

Therefore we can prove Theorem 4 by showing that

$$|\mathbf{v}| \int_{-\infty}^{\infty} \sum_{k=2}^n (\sin(\tfrac{1}{2} v_k t)) / (\tfrac{1}{2} v_k t) dt \quad (17)$$

is less than 10π .

To do this we divide the range of integration into several pieces and bound the contribution to the total of each piece. The first interval is $[0, 4/v_2]$. Subsequent intervals are of the form $[4/v_k, 4/v_{k+1}]$ and the last interval is $[4/v_{k^*}, \infty]$ where $2 \leq k^* \leq n$ is still to be determined. Let $K = \frac{1}{6}$. Then $(\sin u)/u < \exp(-Ku^2)$ for $0 < u \leq 2$, and of course $(\sin u)/u < \frac{1}{2}$ for $u > 2$.

On $[0, 4/v_2]$ the integrand in (17) is less than $\exp(-\frac{1}{4} Kt^2(|\mathbf{v}|^2 - 1))$ so the

first piece contributes to (17) less than

$$|v| \cdot \frac{1}{2} \cdot (2\pi)^{1/2} \sqrt{12} \cdot (|v|^2 - 1)^{-1/2} < 5.$$

On the intervals $[4/v_k, 4/v_{k+1}]$ the integrand of (17) is less than

$$P_k(t) = \prod_{j=2}^k \frac{1}{2} \prod_{j=k+1}^n \exp\left(-\frac{1}{4} K v_j^2 t^2\right). \quad (18)$$

Let $r_k^2 = \frac{1}{4} \sum_{j=k+1}^n |v_j|^2$. Then $P_k(t) = (\frac{1}{2})^{k-1} \exp(-K r_k^2 t^2)$. For $t > 4/v_k$, $t^2 > 4^2/v_k^2 + 8(t - 4/v_k)/v_k$. Thus

$$\int_{4/v_k}^{\infty} P_k(t) dt < \left(\frac{1}{2}\right)^{k-1} \exp\left(-\frac{8}{3} r_k^2 / v_k^2\right) \cdot 3 v_k / 4 r_k^2.$$

Let k^* be the least k such that $r_k^{-1} \geq (\frac{3}{2}) r_{k-1}^{-1}$. Then for $k < k^*$, $r_k^2 \geq \frac{1}{5} v_k^2$, while $r_{k^*}^2 \leq \frac{1}{5} v_{k^*}^2$. Thus for $k < k^*$,

$$\int_{4/v_k}^{\infty} P_k(t) dt < \left(\frac{1}{2}\right)^{k-1} \cdot \frac{1}{r_k}. \quad (19)$$

Now $(\frac{1}{2})^{k-1} r_k^{-1} < \frac{3}{4} (\frac{1}{2})^{k-2} r_{k-1}^{-1}$ for $k < k^*$, so that the sum for $k < k^*$ of the bounds of (19) is less than $4 \cdot r_2^{-1} < 5|v|^{-1}$, and the total contribution to (17) from $[-4/v_{k^*}, 4/v_{k^*}]$ is less than $2(5 + 5) = 20$.

The integrand of (17) on $[4/v_{k^*}, \infty]$ is less than

$$2^{k^*-1} \prod_{j=2}^{k^*} v_j^{-1} t^{1-k^*}.$$

Now by the definition of k^* ,

$$v_{k^*-1}^2 \leq \frac{5}{4} v_{k^*}^2 + \frac{5}{4} \left(\sum_{j=k+1}^n v_j^2 \right) \leq \frac{9}{4} v_{k^*}^2.$$

By induction we have similarly

$$v_{k^*-m}^2 \leq \left(\frac{9}{4}\right)^m v_{k^*}^2, \quad (20)$$

so that on summing $|v|^2 < (\frac{9}{4})^{k^*} v_{k^*}^2$ and $v_{k^*} > (2/3)^{k^*} |v|$. The contribution to (17) from $[4/v_{k^*}, \infty]$ is thus less than

$$\begin{aligned} & |v| (k^* - 2)^{-1} 4^{2-k^*} 2^{k^*-1} v_{k^*}^{-1} \prod_{j=2}^{k^*} (v_{k^*} v_j^{-1}) \\ & < \left(\frac{3}{2}\right)^{k^*} v_{k^*} (k^* - 2)^{-1} 4^{2-k^*} 2^{k^*-1} v_{k^*}^{-1} \prod_{j=2}^{k^*} \left(\frac{2}{3}\right)^{k^*-j} < 1. \end{aligned}$$

Thus in all, the expression in (17) is less than 22, which is less than 10π . $\square$

REMARK. I conjecture that $M = \sqrt{2}$ is possible in Theorem 4. If so, it is clearly the best possible.

BIBLIOGRAPHY

1. Z. W. Birnbaum, *On random variables with comparable peakedness*, Ann. Math. Statist. **19** (1948), 76-81.
2. L. Breiman, *Probability*, Addison-Wesley, Reading, Mass., 1968, Chapter 3.
3. W. Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1966, p. 140.
4. S. Sherman, *A theorem on convex sets with applications*, Ann. Math. Statist. **26** (1955), 763-766.

DEPARTMENT OF MATHEMATICS, TEXAS A & M UNIVERSITY, COLLEGE STATION, TEXAS 77843