ON THE SUCCESSIVE DERIVED SETS OF THE PISOT NUMBERS

DAVID W. BOYD

ABSTRACT. Let S denote the set of Pisot (or Pisot-Vijayaraghavan) numbers and let $S^{(k)}$ be the k th derived set of S . It is shown that $k^{1/2} < \min S^{(k)}$ and that $\limsup(\min S^{(k)}/k) < 1$. The lower bound improves the estimate $k^{1/4} < \min S^{(k)}$ of Dufresnoy and Pisot, while the upper bound improves the obvious estimate $\min S^{(k)} < k + 1$.

The set S of Pisot (or Pisot-Vijayaraghavan) numbers is the set of real algebraic integers $\theta > 1$ all of whose remaining conjugates lie strictly within the unit circle. Salem [6] proved the remarkable fact that S is a closed subset of the real line.

Let $S^{(k)}$ denote the k th derived set of S , and $\min S^{(k)}$ its smallest element. It was shown by Dufresnoy and Pisot [2] that $\min S^{(k)} > k^{1/4}$, and it is known that the integer $k + 1$ is in $S^{(k)}$ [5, p. 57] so $\min S^{(k)} \leq k + 1$.

Our purpose here is to improve these results and to show that in fact $k^{1/2} < \min S^{(k)} < g(k)$ where $\limsup g(k)/k < 1$. It is believed that the lower bound is closer to the truth.

THEOREM 1. $\min S^{(k)} \geq k^{1/2}$.

PROOF. Let C be the set of rational functions $A(z)/Q(z) = f(z)$ for which A and Q have integer coefficients, $Q(0) = 1$, Q has exactly one zero θ^{-1} in $|z| \leq 1$, and $|f(z)| \leq 1$ on $|z| = 1$. Thus θ is in S , and f is said to be associated with θ . If $f(z) = u_0 + u_1z + \dots$, then it is shown in [3] that, for $n \geq 3$, there are polynomials D_n, D_n^* of degree n such that if

$$E_n(z) = -z^n D_n(z^{-1}) \quad \text{and} \quad E_n^*(z) = z^n D_n^*(z^{-1}),$$

then the Taylor expansions of $D_n(z)/E_n(z)$ and $D_n^*(z)/E_n^*(z)$ begin with the terms $u_0 + \dots + u_{n-1}z^{n-1}$, and furthermore, if the next terms are $w_n z^n$ and $w_n^* z^n$ respectively, then $w_n < u_n < w_n^*$. Also D_n, D_n^* have exactly one zero each in $|z| > 1$ at the points τ_n, τ_n^* where $\tau_n < \theta < \tau_n^*$ and $\tau_n \rightarrow \theta, \tau_n^* \rightarrow \theta$ as $n \rightarrow \infty$.

A result of Grandet-Hugot [4] states that if θ is in $S^{(k)}$ then there is an f in

Received by the editors March 10, 1978.

AMS (MOS) subject classifications (1970). Primary 12A15; Secondary 30A42.

Key words and phrases. Pisot number, derived set.

© 1979 American Mathematical Society
0002-9939/79/0000-0052/\$01.75

C associated with θ such that

$$w_n + k \leq u_n \leq w_n^* - k \quad (1)$$

for all sufficiently large n .

Consider now the function

$$\phi(z) = \{(D_n^*/E_n^*) - (D_n/E_n)\}(1 - \tau_n^*z)(\tau_n^* - z)^{-1}(1 - \tau_n z)(\tau_n - z)^{-1}z^{-n}.$$

Clearly $\phi(z)$ is holomorphic in $|z| < 1$, and $|\phi(z)| \leq 2$ on $|z| = 1$ since $|D_n^*/E_n^*| = 1$ and $|D_n/E_n| = 1$ on $|z| = 1$. Thus

$$(w_n^* - w_n)/(\tau_n^* \tau_n) = \phi(0) \leq 2. \quad (2)$$

By (1) and (2), $2k \leq w_n^* - w_n \leq 2\tau_n^* \tau_n$. Letting $n \rightarrow \infty$, we have $k \leq \theta^2$, which proves the theorem.

THEOREM 2. Let $\theta = n$, a rational integer. Then n is in $S^{(k)}$, where

$$k \geq \max\{2[\alpha(n-1)], 2[\alpha(n-2)] + 1\},$$

α being the constant $1/(2\sqrt{2} - 1) > 1/2$. In particular

$$\limsup(\min S^{(k)}/k) \leq 1/(2\alpha) < 1.$$

PROOF. Let $P(a, b, c; z) = z^a + z^b + z^{a+c} - z^{b+c}$ for integers a, b, c . Then

$$|P(a, b, c; z)|^2 = |z^a + z^b|^2 + (z^c + z^{-c})(z^{a-b} - z^{b-a}) + |z^a - z^b|^2 \leq 8,$$

for $|z| = 1$, since we can combine the first and last terms and use the parallelogram law. Thus, if $l-1 \geq 2\sqrt{2}m$ and $n_1, \dots, n_{3m}$ are arbitrary positive integers, then the following polynomial has exactly one zero in $|z| < 1$:

$$1 - lz + P(n_1, n_2, n_3; z) + \dots + P(n_{3m-2}, n_{3m-1}, n_{3m}; z).$$

If we specialize $n_{3i-2} = 1$ for $1 \leq i \leq j$, where $j \leq m$, we obtain a polynomial with $3m - j$ parameters whose root inside the unit circle tends to $1/(l-j)$ as these parameters tend to ∞ . This shows that $n = l-j \in S^{(3m-j)}$ if $l-1 \geq 2\sqrt{2}m$. The most favourable choice of j is $j = m$, giving $n \in S^{(2m)}$ if $n-1 \geq (2\sqrt{2}-1)m$, or else $j = m-1$, giving $n \in S^{(2m+1)}$ if $n-2 \geq (2\sqrt{2}-1)m$. These combine to prove the theorem.

REMARKS. 1. In [1], by a fairly lengthy analysis, we obtain an explicit formula for $\lim(w_n^* - w_n) = 2w(f)$, namely $w(f) = \theta^2 \mathcal{G}(1 - |f|^2)$, where $\mathcal{G}$ denotes the geometric mean on the unit circle. Using this in place of (2) one can improve the lower bound on $\min S^{(k)}$ to $\min S^{(k)} \geq (k+1)^{1/2}$.

2. It is known that $\min S = \theta_0 = 1.3247 \dots$ the real root of $z^3 - z - 1$, that $\min S^{(1)} = (1 + 5^{1/2})/2$ [3], and $\min S^{(2)} = 2$ [4]. No other explicit values of $\min S^{(k)}$ are known.

ACKNOWLEDGEMENT. This work was supported in part by the Canadian National Research Council and a Senior Research Fellowship from the I. W. Killam Foundation.

REFERENCES

1. D. W. Boyd, *Pisot numbers and the width of meromorphic functions* (privately circulated manuscript).
2. J. Dufresnoy and Ch. Pisot, *Sur les dérivés successifs d'un ensemble fermé d'entiers algébriques*, Ann. Sci. École Norm. Sup. (3) **70** (1953), 105–133.
3. ———, *Étude de certaines fonctions méromorphes bornées sur le cercle unité. Application à un ensemble fermé d'entiers algébriques*, Ann. Sci. École Norm. Sup. (3) **72** (1955), 69–92.
4. M. Grandet-Hugot, *Ensembles fermés d'entiers algébriques*, Ann. Sci. École Norm. Sup. (3) **82** (1965), 1–35.
5. Ch. Pisot, *Quelques aspects de la théorie des entiers algébriques*, Séminaire de Mathématiques Supérieures, Université de Montréal, 1966.
6. R. Salem, *A remarkable class of algebraic integers. Proof of a conjecture of Vijayaraghavan*, Duke Math. J. **11** (1944), 103–108.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, B.C., CANADA V6T 1W5