KAROUBI-VILLAMAYOR K -THEORY IS NOT HOMOTOPY QUILLEN THEORY

JAN R. STROOKER

ABSTRACT. It is shown that in general the Karoubi-Villamayor group K^{-2} is not a homomorphic image of Milnor's K_2 , establishing the title statement.

A functor F from rings (possibly without identity) to groups or abelian groups is called a homotopy functor if the canonical map $F(R) \rightarrow F(R[t])$ is an isomorphism for every ring R . To each F one may attach its "homotopization" $HF = \bar{F}$ which is a homotopy functor. There is an epimorphism $F \rightarrow \bar{F}$ (onto for every ring) through which every morphism from F to homotopy functors uniquely factors. For details consult [3], [7] or [6]. Since there are natural transformations from the Quillen-Swan-Gersten K_n -groups to the Karoubi-Villamayor K^{-n} -groups, and the latter are homotopy functors, it is natural to wonder whether they are the homotopizations of the K_n . This is lent credence by the fact that $K^{-1} = \bar{K}_1$ and that for regular rings one even has $K^{-n} = K_n$. The following question was therefore raised [7], [2]:

(Q) Are the canonical morphisms of functors $\bar{K}_n \rightarrow K^{-n}$ isomorphisms for all $n \geq 1$?

These maps can also be described as edge homomorphisms in a spectral sequence due to Gersten and Anderson [4], [1].

While we were working on [6], Sharma and I certainly kept (Q) in mind, but could not reach a conclusion. We did however construct a "bad" ring, which demolished another plausible question concerning Karoubi-Villamayor theory. Once Quillen had, shortly thereafter, proved the so-called Fundamental Theorem of algebraic K -theory, it was clear that our ring also furnished a counterexample to (Q). We announced this in [9] (the "Autorreferat" of [6]); belatedly, the present note fills in the details.

In explaining the example, I shall adopt the point of view, and freely use the results, of [6]. Let R be a ring with identity. The Fundamental Theorem [5, p. 237] asserts there is a split exact sequence

$$0 \rightarrow K_n(R) \rightleftarrows K_n(R[t]) \oplus K_n(R[t^{-1}]) \rightarrow K_n(R[t, t^{-1}]) \rightleftarrows K_{n-1}(R) \rightarrow 0$$

Received by the editors April 3, 1978.

AMS (MOS) subject classifications (1970). Primary 18F25, 13D15.

Key words and phrases. Karoubi-Villamayor and Quillen K -theory, Milnor's K_2 , homotopy functors, Fundamental Theorem.

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0002-9939/79/0000-0054/\$01.50

for $n > 1$. Apply the homotopization functor H and bear in mind that $HK_n = \bar{K}_n$ is a homotopy functor. This gives a split short exact sequence

$$0 \rightarrow \bar{K}_n(R) \rightarrow \bar{K}_n(R[t, t^{-1}]) \rightarrow \bar{K}_{n-1}(R) \rightarrow 0.$$

Now abbreviate the Laurent polynomial functor $-[t, t^{-1}]$ by Λ and the loop space functor $(t^2 - t) -$ by Ω and write K^0 for $\bar{K}_0$. Then $K^{-n}(\Lambda R) = K^{-1}(\Omega^{n-1}\Lambda R) = K^{-1}(\Lambda\Omega^{n-1}R)$. The latter group equals $K^{-1}(\Omega^{n-1}R) \oplus K^0(\Omega^{n-1}R)$ by [6, p. 88], hence $K^{-n}(R) \oplus K^0(\Omega^{n-1}R)$. By the universality of H , we get compatible vertical maps between the two split short exact sequences:

$$\begin{array}{ccccccc} 0 & \rightarrow & \bar{K}_n(R) & \rightarrow & \bar{K}_n(R[t, t^{-1}]) & \rightarrow & \bar{K}_{n-1}(R) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & K^{-n}(R) & \rightarrow & K^{-n}(R[t, t^{-1}]) & \rightarrow & K^0(\Omega^{n-1}R) \rightarrow 0 \end{array}$$

If both maps are isomorphisms, so will be the induced map $\bar{K}_{n-1}(R) \rightarrow K^0(\Omega^{n-1}R)$. For $n = 2$ we have $\bar{K}_1 = K^{-1}$, so it is enough to find a ring R with $K^{-1}(R) \cong K^0(\Omega R)$. Such a ring was exhibited in [6, p. 93]: $R = \mathbb{Z}[X]/(X^2 - 2^{2m})$, $m > 1$. In fact $K^{-1}(R) = \mathbb{Z}/(2)$ while $K^0(\Omega R) = \mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$. Therefore $K^{-2}(A)$ is not a homomorphic image of $K_2(A)$ for $A = \mathbb{Z}[X]/(X^2 - 2^{2m})[t, t^{-1}]$, $m > 1$, and $\bar{K}_2(A) \rightarrow K^{-2}(A)$ is not surjective.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTRECHT, 3508 TA UTRECHT, THE NETHERLANDS