TWO APPLICATIONS OF CHANGE OF RINGS THEOREMS FOR POINCARÉ-SERIES

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ABSTRACT. Let (R, m, k) be an artinian Gorenstein ring with $\dim_k m/m^2 > 2$. If P^R denotes the Poincaré-series and ϕ_R denotes the Bass-series of R , then $\phi_R = (P^R - 1)X^{-1}(1 - P^R X^2)^{-1}$ with $\bar{R} = R/0$; m , see Proposition 1. With similar methods as in the proof of Proposition 1 we calculate the Poincaré-series of a certain class of local rings: Let (R, m, k) be a local ring, (α_{ij}) a $(n-1) \times n$ -matrix with entries in m , $n > 3$. Let $x_1, \dots, x_n$ be elements in m and $\mathfrak{A}$ the ideal generated by the maximal minors of (α_{ij}) and by $\sum_j \alpha_{ij} x_j$ ($i = 1, \dots, n-1$). If $\text{depth } \mathfrak{A} = n$, then $P^{R/\mathfrak{A}} \cdot ((1 - X)^{n-1} - X)(1 + X)^n = P^R$.

Throughout this paper all rings are assumed to be noetherian and local, and all modules are assumed to be of finite type. Let (R, m, k) be a local ring and let M be an R -module. The Poincaré-series of M is defined to be

$$P_M^R = \sum_{i \geq 0} \dim_k \text{Tor}_i^R(k, M) X^i.$$

If M coincides with k we simply write P^R instead of P_k^R .

It is conjectured that P_M^R is a rational function. By a theorem of Gulliksen this problem can be reduced to the case $M = k$.

THEOREM 1 (GULLIKSEN [3, THEOREM 2]). *Let $R \ltimes M$ denote the trivial extension of R by an R -module M . Then*

$$(P^{R \ltimes M})^{-1} = (P^R)^{-1} (1 - P_M^R \cdot X).$$

In the present paper we use this result to derive a formula for the Bass-series

$$\phi_R = \sum_{i \geq 0} \dim_k \text{Ext}_R^i(k, R) X^i,$$

for a certain class of local rings.

Let (R, m, k) be an artinian Gorenstein ring with $\dim_k m/m^2 > 2$. The socle $0: m$ of R is a k -vector space of dimension 1. Let $\sigma \in 0: m$, $\sigma \neq 0$, and let $\bar{R} = R/\sigma R$.

PROPOSITION 1. $\phi_{\bar{R}} = (P^R - 1)X^{-1}(1 - P^R \cdot X^2)^{-1}$.

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If for instance R is an artinian complete intersection with $\dim_k m/m^2 = n$, $n > 1$, then

$$\phi_{\bar{R}} = (1 - (1 - X)^n)(X(1 - X)^n - X^3)^{-1}.$$

Since $P^R = (1 - X)^{-n}$ [9, Theorem 6] the formula follows from Proposition 1.

To prove the proposition we need a second result.

THEOREM 2 (AVRAMOV-LEVIN [6, Theorem 2.9]). *With the assumptions and notations of Proposition 1 the following formula holds: $P_{\bar{R}}^{-1} = P_R^{-1} - X^2$.*

Gulliksen [3, Theorem 1] proved a similar result in the case that R is a complete intersection. Theorem 2 can also be found in [5, Satz 4.7], however with some restrictive assumptions on the homology algebra of the Koszul complex of R .

PROOF OF PROPOSITION 1. Let E denote an injective hull of the residue class field k of R . We observe the following simple facts:

$$(1) \phi_{\bar{R}} = P_E^{\bar{R}}.$$

$$(2) E \simeq \text{Hom}_R(\bar{R}, R) \simeq m.$$

This means that m considered as an $\bar{R}$ -module is an injective hull of k .

(3) $\bar{R} \ltimes m$ is a Gorenstein ring, whose socle is generated by $(0, \sigma)$ [3, Lemma to Theorem 3].

Now let S denote the residue class ring $\bar{R} \times m/(0, \sigma)\bar{R} \times m$. Using Theorems 1 and 2 we find

$$\begin{aligned} (P^S)^{-1} &= (P^{\bar{R} \ltimes m})^{-1} - X^2 = (P^{\bar{R}})^{-1} (1 - \phi_{\bar{R}} \cdot X) - X^2 \\ &= ((P^R)^{-1} - X^2)(1 - \phi_{\bar{R}} \cdot X) - X^2. \end{aligned} \quad (4)$$

We have $S \simeq \bar{R} \ltimes \bar{m}$, where $\bar{m}$ denotes the maximal ideal of $\bar{R}$.

Using again Theorems 1 and 2 we get

$$\begin{aligned} (P^S)^{-1} &= (P^{\bar{R}})^{-1} (1 - P_{\bar{m}}^{\bar{R}} X) \\ &= (P^{\bar{R}})^{-1} (2 - P^{\bar{R}}) = 2(P^R)^{-1} - 2X^2 - 1. \end{aligned} \quad (5)$$

Comparing (4) and (5) the result follows.

As a second application of Theorems 1 and 2 we show

THEOREM 3. *Let (R, m) be a local ring, $n > 3$, (α_{ij}) a $(n-1) \times (n-1)$ -matrix with entries in m ; $x_1, \dots, x_n$ elements in m , $\mathfrak{A}$ the ideal generated by the maximal minors of (α_{ij}) and by $\sum_j \alpha_{ij} x_j$ ($i = 1, \dots, n-1$). If $\text{depth}(\mathfrak{A}) = n$, then*

$$P^{R/\mathfrak{A}}(X) \cdot ((1 - X)^{n-1} - X)(1 + X)^n = P^R(X).$$

PROOF. Let I be the ideal of $\mathbb{Z}[X_j, Y_{ij}]_{i=1, \dots, n-1; j=1, \dots, n}$ generated by the maximal minors of (Y_{ij}) and by $\sum_j Y_{ij} X_j$ ($i = 1, \dots, n-1$). Since I is generically perfect of depth n , see [4, Corollary 4.5, Example 3], we obtain,

using Theorem 6.2 in [2], that it suffices to prove

$$P^{\tilde{R}/\tilde{\mathfrak{A}}}(X) = (1 + X)^{n^2-n} / ((1 - X)^{n-1} - X),$$

where $\tilde{R}$ denotes the ring of formal power series $k[X_j, Y_{ij}]_{i=1, \dots, n-1, j=1, \dots, n}$ with k the residue class field of R , and $\tilde{\mathfrak{A}}$ the ideal in $\tilde{R}$ generated by the maximal minors of (Y_{ij}) and by $\sum_j Y_{ij} X_j$ ($i = 1, \dots, n-1$). Dividing $\tilde{R}/\tilde{\mathfrak{A}}$ by the sequence S of length $n^2 - n$ of the residue classes of

$$\begin{aligned} Y_{ij} \quad (j \neq 1, i \neq j-1), \quad Y_{i,1} - X_1 \quad (i = 1, \dots, n-1), \\ Y_{jj+1} + X_{j+1} \quad (j = 1, \dots, n-1) \end{aligned}$$

we obtain the 0-dimensional ring $A = k[X_1, \dots, X_n]/\mathfrak{C}$ where $\mathfrak{C}$ denotes the ideal generated by $X_1^2 - X_2^2, \dots, X_{n-1}^2 - X_n^2$ and $X_1 \cdots \hat{X}_i \cdots X_n$ ($i = 1, \dots, n$) ($\hat{X}_i$ is omitted in the product $X_1 \cdots \hat{X}_i \cdots X_n$).

Since $\tilde{R}/\tilde{\mathfrak{A}}$ is a Cohen-Macaulay ring of dimension $n^2 - n$, in fact it is even Gorenstein, see [4, §4, Example 3], and $\dim A = 0$, the sequence S is a regular $\tilde{R}/\tilde{\mathfrak{A}}$ -sequence and we get $P^A(X)(1 + X)^{n^2-n} = P^{\tilde{R}/\tilde{\mathfrak{A}}}(X)$. So it is sufficient to prove

$$P^A(X) = 1 / ((1 - X)^{n-1} - X).$$

Let $x_1, \dots, x_n$ be the residue classes of $X_1, \dots, X_n$ in A . The socle of A is generated by $\sigma = x_1^{2n-4} (= x_i^{2n-4} \text{ for } i \in \{2, \dots, n\})$. Let σ_i denote the element $x_i x_j^{2n-6}$ then $x_i \sigma_i = \delta_{i,l} \sigma$ ($\delta_{i,l}$ Kronecker-delta). Let $K = A\langle T_1, \dots, T_n; dT_i = x_i \rangle$ be the Koszul complex of A . We will now define a set of cycles in K , which represents a k -basis for the maximal ideal $\tilde{H}(K)$ of the homology algebra $H(K)$.

Let us first consider indices I of the form $I = (i_1, \dots, i_l)$ with $1 \leq i_1 < \dots < i_l \leq n$. We write $i_1, \dots, i_l \in I$ and $|I| = l$. Let I, J be two such indices. By $I \subseteq J$ we mean that $i \in I$ implies $i \in J$. By $I \cup J$ resp. $I \cap J$ we denote the index $(k_1, \dots, k_s)$, where $\{k_1, \dots, k_s\}$ is the union resp. intersection of $\{i_1, \dots, i_l\}$ and $\{j_1, \dots, j_r\}$, if $I = (i_1, \dots, i_l)$ and $J = (j_1, \dots, j_r)$. For $I = (i_1, \dots, i_l)$ and $j = i_s$ for some $1 \leq s \leq l$, I_j denotes the index $(i_1, \dots, \hat{i}_s, \dots, i_l)$ with i_s omitted. $I_{j,r}$ denotes $(I_j)_r$. Let $I \neq (1, \dots, n)$ be an index; the complementary index CI is the index, so that $I \cap CI = \emptyset$ and $I \cup CI = (1, \dots, n)$.

Now we define some elements in K . Let $I = (i_1, \dots, i_l) \subseteq (1, \dots, n)$ be an index, then:

$$\begin{aligned} x_I &:= x_{i_1} \cdots x_{i_l}; x_\emptyset = 1, T_I := T_{i_1} \cdots T_{i_l}, \\ Y_I &:= x_I T_I, \\ Z_I &:= x_{I_1} T_{i_1}, Z_I^{(j)} := x_{I_{j,j}} T_{i_j} T_j, \text{ if } j \in I, j \neq i_1, \\ u_{(i)} &:= u_i := Y_{(1)} - Y_{(i)}, i \in \{2, \dots, n\}, \\ u_I &:= u_{i_1} \cdots u_{i_l}, I \subseteq (2, \dots, n), \\ w_I &:= (\sum_{j=1}^l (-1)^{j+1} Y_{I_j}) Z_{CI}, I \neq (1, \dots, n), \\ v_i &:= (-1)^i x_{(2, \dots, n)_i} T_{(1,i)}, i \in \{2, \dots, n\}, \\ v_I &:= u_{I_i} v_{i_l}, I \subseteq (2, \dots, n). \end{aligned}$$

We claim that the elements

$$u_I, \quad I \subseteq (2, \dots, n), \quad I \neq (2, \dots, n),$$

$$w_I, \quad I \subseteq (1, \dots, n), \quad I \neq (1, \dots, n),$$

$$v_I, \quad I \subseteq (2, \dots, n)$$

represent a k -basis of $\tilde{H}(K)$.

One immediately sees that these elements are cycles. Let $I, I', J, J' \subseteq (2, \dots, n)$; $I, I' \neq (2, \dots, n)$, $L, L' \subseteq (1, \dots, n)$; $L, L' \neq (1, \dots, n)$, then:

$$u_I u_{I'} = \begin{cases} 0, & \text{if } I \cap I' \neq \emptyset \text{ or if } I \cup I' = (2, \dots, n), \\ \pm u_{I \cup I'}, & \text{otherwise.} \end{cases}$$

$$u_I v_J = \begin{cases} 0, & \text{if } I \cap J \neq \emptyset, \\ \pm v_{I \cup J}, & \text{otherwise.} \end{cases}$$

$$u_I w_L = \begin{cases} 0, & \text{if } I \cap L \neq \emptyset, \\ (-1)^{|I|+|L|} d(u_{I_1} Y_L Z_{CL}^{(i_1)}), & \text{otherwise } (I = (i_1, \dots)). \end{cases}$$

$$v_J v_{J'} = 0, \quad v_J w_L = 0.$$

$$w_L w_{L'} = \begin{cases} 0, & \text{if } L \cap L' \neq \emptyset, \\ \pm v_{(2, \dots, n)}, & \text{if } L \cap L' = \emptyset \text{ and } L \cup L' = (1, \dots, n), \\ d(\sigma_r T_r T_L T_{L'}), & \text{otherwise } (C(L \cup L') = (v, \dots)). \end{cases} \quad (*)$$

Since A is a 0-dimensional Gorenstein ring, $H(K)$ is a Poincaré algebra. Especially

$$\dim H_i(K) = \begin{cases} 1, & \text{if } i = 0, n, \\ 2n - 1, & \text{if } i = 1, n - 1, \\ 2\binom{n}{i}, & \text{if } i \in \{2, \dots, n - 2\}, \end{cases}$$

see [4, Corollary 4.6].

The socle in $H(K)$ is obviously generated by $v_{(2, \dots, n)}$. For every $l \in \{1, \dots, n - 1\}$ the multiplication $K_l \times K_{n-l} \rightarrow K_n$ is a bilinear map and for $u_I, v_J, w_L \in K_l$ and $u_{I'}, v_{J'}, w_{L'} \in K_{n-l}$ one has

$$u_I u_{I'} = u_I w_{L'} = 0; \quad u_I v_{J'} = \begin{cases} \pm v_{(2, \dots, n)}, & \text{if } J' = (CI)_l, \\ 0, & \text{otherwise.} \end{cases}$$

$$v_J v_{J'} = v_J w_{L'} = 0; \quad v_J u_{I'} = \begin{cases} \pm v_{(2, \dots, n)}, & \text{if } I' = (CJ)_l, \\ 0, & \text{otherwise.} \end{cases}$$

$$w_L u_{I'} = w_L v_{J'} = 0; \quad w_L w_{L'} = \begin{cases} \pm v_{(2, \dots, n)}, & \text{if } L' = CL, \\ 0, & \text{otherwise.} \end{cases}$$

Since these equations are also valid modulo boundaries, the residue classes of u_I, v_J , and w_L are linearly independent in $H_l(K)$ and since $\#(\{u_I, v_J, w_L\} \cap K_l) = \dim H_l(K)$, $\{u_I, v_J, w_L\} \cap K_l$ represents a basis for $H_l(K)$.

Massey operations in the sense of Shamash [7, §4] are given on these representatives of a basis of $\tilde{H}(K)$ by:

$$\begin{aligned}\gamma(u_I, u_{I'}) &= \gamma(u_I, v_J) = \gamma(v_J, v_{J'}) = \gamma(v_J, w_L) = 0. \\ \gamma(u_I, w_L) &= \begin{cases} 0, & \text{if } I \cap L \neq \emptyset, \\ (-1)^{|I|+1} u_{I_i} Y_L Z_{CL}^{(i)}, & \text{otherwise } (I = (i_1, \dots)). \end{cases} \\ \gamma(w_L, w_{L'}) &= \begin{cases} 0, & \text{if } L \cap L' \neq \emptyset, \text{ or } L \cup L' = (1, \dots, n), \\ \sigma_\nu T_\nu T_L T_{L'}, & \text{otherwise } (C(L \cup L') = (\nu, \dots)), \end{cases}\end{aligned}$$

for $I, I', J, J' \subseteq (2, \dots, n)$, $I, I' \neq (2, \dots, n)$ and $L, L' \subseteq (1, \dots, n)$; $L, L' \neq (1, \dots, n)$.

One now checks that all Massey operations of higher orders vanish in K .

Applying Avramov's Theorems 5.1 and 5.3 in [1], we obtain

$$P^A(X) = (1 + X)^n (P^H(X, X))^{-1}. \quad (1)$$

So the problem to compute P^A is reduced to compute P^H . One now can check that H with a suitable arrangement of the u_I, v_J, w_L satisfies the hypothesis in §2 in [8]. Therefore one can compute the Poincaré-series P_k^H with methods Shamash developed in his papers [7], [8] using a formula stated in [5, §4]. However, there is a more simple way of computing. Let $\bar{H}$ be the homology-algebra H reduced by its socle $H_n(K) = k[v_{(2)}, \dots, v_{(n)}]$. By [5, Proof of 4.7], or Theorem 2 (graded case) we have

$$(P^{\bar{H}}(X, Y))^{-1} = (P^H(X, Y))^{-1} + X^2 Y^n. \quad (2)$$

As the formulas (*) show, we have $\bar{H} = \bar{E} \rtimes M$ where E is the exterior algebra over k in $(n-1)$ generators (the residue classes of $u_{(2)}, \dots, u_{(n)}$) of degree 1, and $\bar{E}$ is the reduction of E modulo its socle. Finally $\bar{E} \rtimes M$ denotes the trivial extension of $\bar{E}$ by a module M .

Now $M = V \oplus W$ as E -module. V, W are the submodules generated by the residue classes of the $v_I, I \neq (2, \dots, n)$, resp. w_L . Denoting the maximal ideal of $\bar{E}$ by $\mathfrak{M}$ we have (see (*)) $V \simeq \mathfrak{M}[-1]$ and $\mathfrak{M}W = 0$. The Hilbert function of W as $\bar{E}$ -module is

$$\mathfrak{H}_W(Y) = (1 + Y)^n - 1 - Y. \quad (3)$$

So we obtain

$$P_M^{\bar{E}}(X, Y) = \mathfrak{H}_W(Y) P^{\bar{E}}(X, Y) + X^{-1} Y (P^{\bar{E}}(X, Y) - 1), \quad (4)$$

by a slight generalisation to the graded case of Theorem 1.

$$P^{\bar{E} \rtimes M}(X, Y) = P^{\bar{E}}(X, Y) (1 - P_M^{\bar{E}}(X, Y) X)^{-1}, \quad (5)$$

and by a generalisation of Theorem 2 to the graded case

$$(P^{\bar{E}}(X, Y))^{-1} = (1 - XY)^{n-1} - X^2 Y^{n-1}. \quad (6)$$

Combining the formulas (1)–(6) yields the desired formula for P^A .

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