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MINIMAL PRIMITIVE IDEALS OF GCR C^*-ALGEBRAS

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ABSTRACT. We show that a minimal primitive ideal of a GCR algebra may contain the maximal CCR ideal of the algebra, thus giving a negative answer to a question of J. Dixmier.

In [**2**, Remarque C, p. 111] (see also [**3**, 4.7.8]), Dixmier has observed that for a GCR C^*-algebra A, any primitive ideal not containing the maximal CCR ideal I of A is in fact a minimal primitive ideal, and he asks whether the converse is true. In terms of the Jacobson topology of the primitive ideal space Prim A, one may rephrase his question as follows (since, by [**3**, 3.2.1, 3.2.2, and 4.7.15], $\{P \in \text{Prim}\, A: P \not\supseteq I\}$ is the maximal open T_1 subset of Prim A): If $P \in \text{Prim}\, A$ is not contained in the closure of any other point of Prim A, must P be contained in the maximal open T_1 subset of Prim A; that is, must P have an open T_1 neighborhood?

We provide a negative answer to this question by considering the C^*-algebra $C^*(\mathbf{R}, X)$ associated (in the sense of [**4**, p. 890]) to a transformation group $(\mathbf{R}, X)$, where X is a certain closed subset of $\mathbf{R}^3$ on which $\mathbf{R}$ acts. $(\mathbf{R}, X)$ is described below by giving the (countably many) orbits into which X is partitioned under $\mathbf{R}$, together with, for each orbit, an $\mathbf{R}$-equivariant map of $\mathbf{R}$ (which acts on itself by $(t, s) \to t + s$) onto the orbit; these maps determine the $\mathbf{R}$-action completely.

There is one family of "vertical" orbits Y_n, $n = 0, 1, \ldots,$ passing through the points $y_0 = (0, 0, 0)$ and $y_n = (2^{-2n}, 0, 0)$ $(n \geqslant 1)$ respectively, with the maps $\mathbf{R} \to Y_n$, $t \to ty_n$ given by

$$ty_0 = (0, t, 0),$$

$$ty_n = (2^{-2n}, t, 0), \qquad n \geqslant 1.$$

There is another family of orbits Z_n, $n = 1, 2, \ldots,$ such that Y_n lies in the closure of Z_n for each $n \geqslant 1$. Z_n passes through $z_n = (2^{-2n+1}, 0, 0)$ and is described by (here m ranges over the integers $\geqslant n + 1$):

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$$tz_n = \begin{cases} (2^{-2n+1}, t, 0), & t \leqslant n, \\ (2^{-2n} + (4 - 3t_m)2^{-2m}, (m-1)\cos t_m\pi, (m-1)\sin t_m\pi), & \\ \quad m^2 - n^2 - n - 1 \leqslant t \leqslant m^2 - n^2 - n + 1, & \\ \qquad \big(\text{where } t_m = \big(t - (m^2 - n^2 - n - 1)\big)/2\big), & \\ (2^{-2n} + 2^{-2m}, t - m^2 - m + n^2 + n, 0), & \\ \quad m^2 - n^2 - n + 1 \leqslant t \leqslant (m+1)^2 - n^2 - n - 1. & \end{cases}$$

Thus Z_n consists of an initial infinite vertical segment, together with a sequence of finite vertical segments, increasing in length and converging to Y_n, which are connected by half circles (each running from the top of one segment to the bottom of the next). Since the half circles increase in diameter as they approach the planes $x = y_n$, $n = 0, 1, \ldots,$ they have no point of accumulation, which ensures that $X = (\bigcup_{n=0}^{\infty} Y_n) \cup (\bigcup_{n=1}^{\infty} Z_n)$ is closed (and hence locally compact); and since points on the vertical pieces are all moved by $\mathbf{R}$ "at the same rate", $(\mathbf{R}, X)$ is jointly continuous.

By [**4**, Theorems 2.1 and 2.2] the C^*-algebra $A = C^*(\mathbf{R}, X)/J$ (where J is the intersection of the kernels of the "induced" representations of $C^*(\mathbf{R}, X)$; although we do not need to know it here, $J = (0)$, by [**6**, Proposition 2.2]) is Type I (and hence GCR, by [**5**]) and Prim A is homeomorphic to the orbit space $X/\mathbf{R}$. Now Y_0 is not contained in the closure of any other point in $X/\mathbf{R}$, but every neighborhood of Y_0 contains points Z_n and Y_n $(\in \{Z_n\}^-)$ and so is not T_1. So the primitive ideal corresponding to $\{Y_0\}$ is minimal but has no T_1 neighborhood of Prim A.

This algebra has CCR composition series of length two. It may be regarded as a simple example of a type of phenomenon, quite different from that considered in [**1**], which one must consider in classifying GCR algebras via the extensions that arise at each step of their CCR composition series.

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