A HELLY-TYPE THEOREM FOR THE DIMENSION OF THE KERNEL OF A STARSHAPED SET

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ABSTRACT. This study will investigate the dimension of the kernel of a starshaped set, and the following result will be obtained: Let S be a compact set in some linear topological space L . For $1 < k < n$, the dimension of $\ker S$ is at least k if and only if for some $\epsilon > 0$ and some n -dimensional flat F^n in L , every $f(n, k)$ points of S see via S a common k -dimensional neighborhood in F^n having radius ϵ . The number $f(n, k)$ is defined inductively as follows:

$$\begin{aligned} f(2, 1) &= 4, \\ f(n, k) &= f(n-1, k) + n + 2 \quad \text{for } 3 \leq n \text{ and } 1 < k \leq n-1, \\ f(n, n) &= n + 1. \end{aligned}$$

1. Introduction. We begin with some preliminary definitions: Let S be a subset of some linear topological space. For points x and y in S , we say x *sees* y via S if and only if the corresponding segment $[x, y]$ lies in S . The set S is said to be *starshaped* if and only if there is some point p in S such that, for every x in S , p sees x via S . The set of all such points p is called the (convex) *kernel* of S , denoted $\ker S$.

A well-known theorem by Krasnosel'skiĭ [3] states that if S is a compact set in some linear topological space L , then S is starshaped if and only if for some n -dimensional flat F^n in L , every $n + 1$ points of S see via S a common point of F^n . It would be interesting to obtain an analogue of Krasnosel'skiĭ's theorem for sets whose kernel is at least k -dimensional, $1 < k < n$. An appropriate analogue concerning the dimension of the kernel of a planar set was given in [1], and in this work, we develop a technique to extend these planar results to higher dimensions.

As in [1], the following familiar terminology will be used: Throughout the paper, $\text{conv } S$, $\text{cl } S$, $\text{int } S$, $\text{rel int } S$, $\text{bdry } S$, and $\ker S$ will denote the convex hull, closure, interior, relative interior, boundary, and kernel, respectively, of the set S . Further, if S is convex, $\dim S$ will represent the dimension of S .

2. Proof of the theorem.

THEOREM. *Let S be a compact subset of some linear topological space L . For $0 \leq k < n$, the dimension of $\ker S$ is at least k if and only if for some $\epsilon > 0$ and some n -dimensional flat F^n in L , every $f(n, k)$ points of S see via S a*

Received by the editors March 17, 1978.

AMS (MOS) subject classifications (1970). Primary 52A30, 52A35; Secondary 52A20.

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0002-9939/79/0000-0068/\$02.00

common k -dimensional neighborhood in F^n having radius ε . The number $f(n, k)$ is defined inductively as follows:

$$f(2, 1) = 4,$$

$$f(n, k) = f(n - 1, k) + n + 2 \quad \text{for } 3 \leq n \text{ and } 1 \leq k \leq n - 1,$$

$$f(n, n) = n + 1.$$

PROOF. The necessity of the condition is obvious, so we need only establish its sufficiency. We consider separately the cases for $k = n$ and $1 \leq k \leq n - 1$, and the proof is accomplished by Lemmas 1 and 2 below.

LEMMA 1. *If every $f(n, n) = n + 1$ points of S see a common n -dimensional neighborhood in F^n having radius ε , then $\ker S$ contains an n -dimensional neighborhood having radius ε .*

PROOF OF LEMMA 1. For convenience of notation, for each y in S we define the set $S_y = \{x: x \in F^n \text{ and } [x, y] \subseteq S\}$, and define $V_y = \{x: x \text{ is the center of an } \varepsilon\text{-neighborhood in } S_y\}$. From our hypothesis, every $n + 1$ of the compact convex sets $\text{conv } V_y$ have a nonempty intersection, so by Helly's theorem in F^n ,

$$\bigcap \{\text{conv } V_y: y \text{ in } S\} \neq \emptyset.$$

Select a point z in $\bigcap \{\text{conv } V_y\} \subseteq \bigcap \{\text{conv } S_y\}$. Using Carathéodory's theorem, since $z \in \text{conv } V_y$ for every y in S , z is in the convex hull of $n + 1$ or fewer points in V_y for each y . Furthermore, since every point of V_y is the center of an ε -neighborhood in S_y , an easy geometric argument reveals that z itself must be the center of an ε -neighborhood in $\text{conv } S_y$ for every y . However, by adapting a standard proof of Krasnosel'skiĭ's theorem [2, Theorem 2.5], we see that

$$\bigcap \{\text{conv } S_y\} = \ker S \cap F^n.$$

Hence z is the center of an ε -neighborhood in $\ker S \cap F^n$, and $\ker S$ contains an n -dimensional ε -neighborhood, the desired result for Lemma 1.

LEMMA 2. *Let $2 \leq n$ and $1 \leq k \leq n - 1$. If every $f(n, k)$ points of S see a common k -dimensional neighborhood in F^n having radius ε , then $\dim \ker S \geq k$.*

PROOF OF LEMMA 2. We induct on the dimension of the flat F^n . If $n = 2$, the number $f(2, 1) = 4$ follows easily from the proof of [1, Theorem 1]. Hence, assume that the lemma is true for natural numbers less than n , $3 \leq n$, to prove for n . Since $n + 1 \leq f(n, k)$, by the proof of Krasnosel'skiĭ's theorem, there is some point z in $\ker S \cap F^n$, and without loss of generality, let $z \in \text{rel int}(\ker S \cap F^n)$. Let D denote the closed n -dimensional ε -ball in F^n centered at z .

By hypothesis, to each $f(n, k)$ -member subset T of S we may associate at least one k -dimensional ε -neighborhood N_T in F^n seen by T . Since $z \in \ker S$, standard arguments reveal that $\text{conv}(N \cup \{t, z\}) \subseteq S$ and hence each point

of T sees $\text{conv}(N \cup \{z\})$ via S . Furthermore, since the radius of N is ε , $\text{bdry } D \cap \text{conv}(N \cup \{z\}) \neq \emptyset$. To each y in S we associate the family $\mathcal{N}_y$, consisting of all k -dimensional ε -neighborhoods N in F^n seen by y . Let $S_y = \{x: x \text{ in } F^n \text{ and } [x, y] \subseteq S\}$ and define $A_y = \{x: x \in D \cap \text{conv}(N \cup \{z\}) \text{ for some } N \text{ in } \mathcal{N}_y\}$. Finally, let $B_y = A_y \cup (\ker S \cap D)$. Clearly $B_y \subseteq S_y$ for each y , so $\cap \{\text{conv } B_y\} \subseteq \cap \{\text{conv } S_y\}$. Also, by the proof of Krasnosel'skii's theorem, $\cap \{\text{conv } S_y\} \subseteq \ker S$, so $\cap \{\text{conv } B_y\} \subseteq \ker S$.

If for some fixed $r > 0$, each set $\text{conv } B_y$ contains an n -dimensional r -neighborhood of z , then $\dim \ker S \geq n$ and the proof is finished. Otherwise, by our choice of z , clearly $\dim(\ker S \cap F^n) < n$. For each y in S we select a hyperplane H_y in F^n supporting $\text{conv } B_y$ for which the distance $d(y)$ from z to H_y is minimal. Let $(H_y)_1$ and $(H_y)_2$ denote the corresponding open halfspaces in F^n , labeled so that $\text{conv } B_y \subseteq \text{cl}(H_y)_1$. Certainly either some $d(y)$ is zero or some sequence of $d(y)$ numbers converges to zero. Hence there is a corresponding sequence of hyperplanes $\{H_{y_m}\} = \{H_m\}$ (possibly constant) which converges to a hyperplane H containing z . We may label the halfspaces H_1 and H_2 determined by H so that a subsequence of $\{(H_m)_1\}$ converges to H_1 . Hence, without loss of generality, assume that $\{\text{cl}(H_m)_1\}$ converges to $\text{cl } H_1$.

Notice that H contains $\ker S \cap F^n$: Since $z \in H \cap \text{rel int}(\ker S \cap F^n)$, either $\ker S \cap F^n \subseteq H$ or H separates points of $\ker S \cap F^n$. If the latter occurred, then for m sufficiently large, H_{y_m} would also separate points of $\ker S \cap F^n$, impossible since $\ker S \cap D \subseteq B_y \subseteq \text{cl}(H_{y_m})_1$ for each y_m .

To complete the proof, we will show that every $f(n-1, k)$ points of S see a common k -dimensional ε -neighborhood in the $(n-1)$ -flat H . First, we establish that every $(f(n, k) - 1)$ -member subset T of S sees a common k -dimensional ε -neighborhood in $\text{cl } H_1$: For each hyperplane H_y and each neighborhood N in $\mathcal{N}_y$, N is necessarily disjoint from $(H_y)_2$. Therefore, each k -dimensional ε -neighborhood seen by $T \cup \{y\}$ necessarily lies in $\text{cl}(H_y)_1$. Since a sequence of these closed halfspaces converges to $\text{cl } H_1$, standard arguments reveal that the points of T see a common k -dimensional ε -neighborhood in $\text{cl } H_1$, and the preliminary result is obtained.

If every $f(n-1, k)$ points of S see a common k -dimensional ε -neighborhood in H , this portion of our argument is complete. Hence we assume that the result fails for some $(f(n-1, k))$ -member subset T' , to reach a contradiction. Since

$$f(n-1, k) = f(n, k) - n - 2 < f(n, k) - 1,$$

by our argument above, points of T' see a common k -dimensional ε -neighborhood in $\text{cl } H_1$. Moreover, if no such neighborhood is in H , then there exists a hyperplane J parallel to H having the following properties: For J_1 one of the corresponding open halfspaces, $\text{cl } J_1 \subseteq H_1$ and every k -dimensional ε -neighborhood in $\text{cl } H_1$ seen by T' contains some point of $\text{cl } J_1$. (Otherwise, using the compactness of S , some sequence of these neighborhoods would converge to a matching neighborhood in H .) Since every $f(n, k) - 1 = f(n-1, k) + n + 1$ points of S see such a neighborhood in $\text{cl } H_1$ and T' has $f(n-1, k)$ members, this forces every $n+1$ points of S to

see a common point of $\text{cl } J_1$. Therefore by the proof of Krasnosel'skiĭ's theorem, $\text{cl } J_1 \cap \ker S \neq \emptyset$. However, this is impossible since $\ker S \cap F^n \subseteq H$ and $\text{cl } J_1 \subseteq H_1$. We have a contradiction, our assumption is false, and we conclude that every $f(n-1, k)$ points of S see a common k -dimensional ε -neighborhood in H , the desired result.

Finally, our inductive hypothesis may be applied to the $(n-1)$ -flat H to conclude that $\dim \ker S \geq k$. This completes the proof of Lemma 2 and finishes the proof of the theorem.

The following example illustrates that the number $f(n, n)$ in the theorem is best possible.

EXAMPLE 1. Let S_1 denote an n -simplex whose interior contains the origin. Define $S_2 = 2S_1$ and let $S = \text{cl}(S_2 \sim S_1)$. Then for an appropriate choice of $\varepsilon > 0$, every n points of S see via S a common n -dimensional ε -neighborhood, yet $\ker S = \emptyset$.

Furthermore, the number $f(2, 1)$ in the theorem is best possible by [1, Example 1]. However, best values for $3 \leq n$ and $1 \leq k \leq n-1$ have not been established.

We conclude with two final remarks:

REMARK 1. Clearly the requirement that S be compact in Theorem 1 may be replaced with the conditions that S be closed and $S \cap F^n$ be compact. However, [1, Examples 3 and 4] may be extended to R^n to show that unless $S \cap F^n$ is both closed and bounded, no finite analogue of the bound $f(n, k)$ is possible.

REMARK 2. The inductive formulas defining $f(n, k)$ in Theorem 1 may be replaced with the following sums:

$$f(n, 1) = \sum \{i: 4 \leq i \leq n+2\} \quad \text{if } 2 \leq n,$$

$$f(n, k) = k+1 + \sum \{i: k+3 \leq i \leq n+2\} \quad \text{if } 2 \leq k \leq n-1,$$

$$f(n, n) = n+1.$$

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