ON SUBVARIETIES OF A HERMITIAN MANIFOLD

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ABSTRACT. We make use of the variational properties of the geodesic distance function of a Riemannian manifold and the technique of the "blowing-up" (σ -process) on a complex manifold to derive the nonexistence of compact complex analytic subvarieties in a simply connected, complete Hermitian manifold with nonpositive sectional curvature.

1. Notation and preliminaries. In this section we introduce the notation and definitions that will be used in the following sections.

(i) Let M^n be a complete connected Riemannian manifold of dimension n and class C^∞ . Throughout this note we will indicate by

r = the geodesic distance function of M .

M_Δ = the diagonal of the product manifold $M \times M$.

M_{cut} = the set of all cut pairs of M .

$M \vee M = M \times M \setminus M_\Delta \cup M_{\text{cut}}$.

$W_c = \{(p, q) \in M \times M / r(p, q) = c\}$, for every positive real number c .

Π_{W_c} = the second fundamental form of W_c (we observe that $W_c \cap M \vee M$ is a submanifold of $M \vee M$ of codimension 1) with $\text{grad } r$ taken as the normal vector to W_c .

For every (p, q) in $M \vee M$ we introduce a natural orthonormal frame of $M \vee M$ at (p, q) as follows:

Let γ be the unique geodesic (parametrized by arc-length) in M between p and q and let $\{e'_i(p), 1 \leq i \leq n\}$ be an orthonormal frame of M at p , with $e'_1(p)$ = the tangent vector to γ at p ; by parallel translation along γ from p to q we construct an orthonormal frame $\{e''_i(q), 1 \leq i \leq n\}$ of M at q and then we set

$$e_{\pm i}(p, q) = \frac{1}{\sqrt{2}} [(e'_i(p), 0) \pm (0, e''_i(q))], \quad 1 \leq i \leq n.$$

We introduce the subspaces

$$V^+_{(p,q)} = \sum_{i=1}^n \mathbf{R} e_i(p, q) \quad \text{and} \quad V^-_{(p,q)} = \sum_{i=1}^n \mathbf{R} e_{-i}(p, q)$$

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and observe that for (p, q) in $M \vee M$

$$T_{(p,q)}W_c = V_{(p,q)}^+ \oplus V_{(p,q)}^- \setminus \mathbf{Re}_{-1}(p, q),$$

where $c = r(p, q)$. Moreover, $e_1(p, q)$ is a principal direction of curvature of W_c at (p, q) with normal curvature equal to zero, see [2].

(ii) Let M be a complex manifold and let p be a point in M . We will denote by $\hat{M}_p$ the "blowing-up" manifold of M at p , by $\sigma: \hat{M}_p \rightarrow M$ the corresponding quadratic transformation and by $B_p = \sigma^{-1}(\{p\})$. Notice that B_p is isomorphic to a complex projective space $\mathbf{CP}^{n-1}$, $n = \dim_{\mathbb{C}} M$. Let V be a complex analytic subvariety of M with p in V , then the set $\hat{V}_p$ (topological closure of $\sigma^{-1}(V \setminus \{p\})$ in $\hat{M}_p$) is a subvariety of $\hat{M}_p$ and it is called the "blowing-up" of the variety V at p .

The projective tangent cone of V at p is defined by

$$K_p(V) = \hat{V}_p \cap B_p.$$

It is known that if V is an irreducible variety of dimension d then $K_p(V)$ is a variety of dimension $d - 1$.

2. Manifolds with nonpositive sectional curvature. In this section we will make use of the variational properties of the geodesic distance function r to derive certain properties about the "position" of the submanifolds W_c when M has nonpositive sectional curvature. We start by proving

PROPOSITION 1. *Let M^n be a simply connected, complete Riemannian manifold with nonpositive sectional curvature. Then for every real number $c > 0$, the submanifold W_c has at every point $2n - 2$ principal directions in which the normal curvature is negative and one principal direction with curvature zero.*

PROOF. Let $(p, q) \in M \vee M$ and set $c = r(p, q)$. Let γ be the unique geodesic (parametrized by arc-length) in M between p and q . For every $v \in (\mathbf{Re}_1(p, q))^\perp$ (the orthogonal complement to $\mathbf{Re}_1(p, q)$ in $T_{(p,q)}W_c$) we construct a variation $W(s, t)$, $0 \leq s \leq c$, $-\infty < t < \infty$, satisfying:

- (a) $W(s, 0) = \gamma(s)$, $0 \leq s \leq c$,
- (b) $W(0, t) = \exp_p(tv_1) = p_t$ and $W(c, t) = \exp_q(tv_2) = q_t$, where $v_1 \in T_p M \setminus \mathbf{Re}'_1(p)$, $v_2 \in T_q M \setminus \mathbf{Re}''_1(q)$ and $v = v_1 + v_2$.
- (c) For each t , $W(t, \cdot)$ is the unique geodesic (parametrized by arc-length) in M between p_t and q_t .

From the fact that M is simply connected and has nonpositive curvature it follows that such variation exists and can be represented as

$$W(s, t) = \exp_{p_t} \left(\frac{sr(p_t, q_t)}{c} \tilde{v}(p_t, q_t) \right)$$

where $\tilde{v}(p_t, q_t) \in T_{p_t}(M)$, $\|\tilde{v}(p_t, q_t)\| = 1$. Observe that $\tilde{v}(p_t, q_t)$ depends differentiably on p_t and q_t . Therefore $W(s, t)$ is a differentiable variation of γ by geodesics with

$$L(W(\cdot, t)) = r(p_t, q_t) = r(\exp_{(p,q)} tv) \quad \text{for all } t, \quad (1)$$

and

$$\left. \frac{d}{dt} L(W(\cdot, t)) \right|_{t=0} = 0. \quad (2)$$

On the other hand the formula for the second variation is [1]

$$\left. \frac{d^2}{dt^2} L(W(\cdot, t)) \right|_{t=0} = \int_0^c (\|Y'\|^2 - K(T, Y)\|Y\|^2 - \langle T, Y' \rangle^2) \Big|_{t=0} ds,$$

where $Y = \partial W / \partial t$, $T = \partial W / \partial s$, $Y' = DY/ds$ (covariant derivative), and $K(T, Y)$ stands for the sectional curvature of the plane spanned by T and Y . We claim that in our case $\langle T, Y' \rangle|_{t=0} = 0$. In fact

$$\begin{aligned} \langle T, Y' \rangle|_{t=0} &= \left\langle \frac{\partial W}{\partial s}, \frac{D}{ds} \frac{\partial W}{\partial t} \right\rangle \Big|_{t=0} \\ &= \left\langle \frac{\partial W}{\partial s}, \frac{D}{dt} \frac{\partial W}{\partial s} \right\rangle \Big|_{t=0} = \frac{1}{2} \frac{\partial}{\partial t} \left\langle \frac{\partial W}{\partial s}, \frac{\partial W}{\partial s} \right\rangle \Big|_{t=0} \\ &= \frac{1}{2} \frac{d}{dt} \langle r(p_t, q_t) \tilde{v}, r(p_t, q_t) \tilde{v} \rangle \Big|_{t=0} = \frac{1}{2} \frac{d}{dt} r^2(p_t, q_t) \Big|_{t=0} \end{aligned}$$

which is zero because of (2). Hence the second variation formula becomes

$$\left. \frac{d^2}{dt^2} L(W(\cdot, t)) \right|_{t=0} = \int_0^c (\|Y'\|^2 - K(T, Y)\|Y\|^2) \Big|_{t=0} ds. \quad (3)$$

Applying (3) to the $2n - 2$ independent directions $\{e_i(p, q) - e_{-i}(p, q); e_{-i}(p, q), 2 \leq i \leq n\}$ we see that for each of them the second variation formula is strictly positive. This, together with the fact that because of (1),

$$\left. \frac{d^2}{dt^2} L(W(\cdot, t)) \right|_{t=0} = \frac{d^2}{dt^2} r(p_t, q_t) \Big|_{t=0}$$

and

$$\left. \frac{d^2}{dt^2} r(p_t, q_t) \right|_{t=0} = -\Pi_{W_c}[v],$$

tells us that W_c has $2n - 2$ principal directions with negative curvature and one principal direction with zero curvature (namely, e_1). This proves the proposition. Now we are able to prove

THEOREM 1. *Let M^n be as in Proposition 1 and let (p, q) in $M \setminus M$, with $c = r(p, q)$. Then there exists $\varepsilon > 0$ sufficiently small, such that*

$$r(p, q) < r(\exp_{(p, q)} v),$$

for all v in $B_\varepsilon(0)$, where $B_\varepsilon(0)$ is the ball with center at 0 and radius ε in $T_{(p, q)} W_c$.

PROOF. Let $f_1, f_2, \dots, f_{2n-1}$ with $f_1 = e_1$ be the orthonormal frame of W_c at (p, q) formed by the principal directions with normal curvatures $k_1 = 0$ and $k_i < 0$, $i \geq 2$. Let us take $\delta > 0$ so small that $\exp_{(p, q)}$ restricted to the ball

$B_\delta(0)$ in $T_{(p,q)}M \times M$ is a diffeomorphism and then set

$$S_\delta = \exp_{(p,q)}(B_\delta(0) \cap T_{(p,q)}W_c).$$

We consider on S_δ coordinates u_i defined by

$$u_i \left[\exp_{(p,q)} \left(\sum_{j=1}^{2n-1} \lambda_j f_j \right) \right] = \lambda_i, \quad 1 \leq i \leq 2n-1.$$

Denote by (p_s, q_s) a point in S_δ satisfying:

$$u_1(p_s, q_s) = s, \quad u_i(p_s, q_s) = 0, \quad i \geq 2.$$

We first prove that

$$dr_{(p,q)}|T_{(p,q)}S_\delta = 0. \quad (4)$$

In fact, for every curve $\alpha(t)$, $|t| < \rho$, in S_δ that goes through the point (p_s, q_s) , i.e. $\alpha(0) = (p_s, q_s)$, we have

$$\alpha(t) = \exp_{(p,q)}v(t), \quad |t| < \rho,$$

where $v(t) \in T_{(p,q)}W_c$.

On the other hand

$$T_{(p,q)}W_c = V_{(p,q)}^+ \oplus V_{(p,q)}^- \setminus \mathbf{Re}_{-1}(p, q),$$

hence

$$v(t) = \lambda_1(t)e_1(p, q) + \sum_{i=2}^n \lambda_i(t)e_i(p, q) + \sum_{j=2}^n \mu_j(t)e_{-j}(p, q).$$

Thus

$$\begin{aligned} \alpha(t) &= \exp_p(\lambda_1(t)e'_1(p) + \text{something perpendicular to } e'_1(p)) \\ &\quad \times \exp_q(\lambda_1(t)e''_1(q) + \text{something perpendicular to } e''_1(q)), \end{aligned}$$

but then

$$\dot{\alpha}(0) = \dot{\lambda}_1(0)e_1(p_s, q_s) + \text{something in } V_{(p,q_s)}^+ \setminus \mathbf{Re}_1 \oplus V_{(p,q_s)}^- \setminus \mathbf{Re}_{-1}$$

which means that

$$T_{(p_s,q_s)}S_\delta \subseteq T_{(p_s,q_s)}W_{r(p_s,q_s)}.$$

By dimensional reasons equality must hold and this proves (4). It follows from (4) that the coordinates $u_1, \dots, u_{2n-1}$ can be written as

$$r(\bar{p}, \bar{q}) = r(p, q) + \frac{1}{2} \sum_{i=2}^{2n-1} k_i u_i^2 + \sum_{i,j=2}^{2n-1} u_i u_j \varphi_{ij}(u)$$

where each $\varphi_{ij}(0) = 0$. Hence there exists $\varepsilon > 0$ sufficiently small such that

$$r(\bar{p}, \bar{q}) \geq r(p, q) \quad \text{for all } (\bar{p}, \bar{q}) \text{ in } S_\varepsilon.$$

This completes the proof of Theorem 1.

Next we apply Theorem 1 to obtain

THEOREM 2. *Let M^n be as in Proposition 1. Then the geodesic distance function r of M^n cannot achieve a positive relative maximum on any k -dimensional ($k > 1$) totally geodesic submanifold V^k in $M \times M$.*

PROOF. Let V^k ($k > 1$) be a totally geodesic submanifold in $M \times M$ and let us assume that r achieves a positive relative maximum c on V^k at (p, q) . We consider through (p, q) the $2n - 1$ dimensional submanifold S_c given by Theorem 1. Then from the definition of S_c we have

$$T_{(p,q)}S_c = T_{(p,q)}W_c \quad (5)$$

and on the other hand, because of the maximality of r on V^k at (p, q) ,

$$T_{(p,q)}V^k \subseteq T_{(p,q)}W_c. \quad (6)$$

From (5) and (6) and the fact that $k > 1$, there exists $0 \neq v \in T_{(p,q)}V^k \setminus \mathbb{R}e_1$ and then by Theorem 1

$$r(p, q) < r(\exp_{(p,q)}tv) \quad \text{for } t \text{ small.} \quad (7)$$

But since V^k is totally geodesic, $\exp_{(p,q)}tv \in V^k$, for all t , and because of the local maximality of r on V^k at (p, q) ,

$$r(p, q) \geq r(\exp_{(p,q)}tv), \quad (8)$$

which contradicts (7). Thus r cannot achieve a positive relative maximum on V^k which was our assertion.

We observe that from Theorem 2 follows the well-known fact that there exists no compact totally geodesic submanifold on a simply connected, complete Riemannian manifold with nonpositive curvature.

3. Hermitian manifolds with nonpositive sectional curvature. In this section we make use of Proposition 1 and the technique developed in [2], to obtain our main result, namely

THEOREM 3. *Let M be a hermitian manifold of complex dimension n , simply connected, complete and with nonpositive sectional curvature. Then $M \vee M$ has no compact complex analytic subvarieties of complex dimension $k > 1$.*

PROOF. Let us assume that there exists V^k a compact complex analytic subvariety of $M \vee M$ ($k > 1$). Then r has a maximum on V^k at (p, q) , $r(p, q) = c$. We know that the tangent space of W_c at (p, q) can be written as

$$T_{(p,q)}W_c = \sum_{i=1}^n \mathbb{R}e_i \oplus \sum_{i=1}^n \mathbb{R}Je_i \oplus \sum_{i=2}^n \mathbb{R}e_{-i} \oplus \sum_{i=1}^n \mathbb{R}Je_{-i}.$$

Now, we consider the following complex subspace of $T_{(p,q)}W_c$,

$$T_{(p,q)}W_c \cap J(T_{(p,q)}W_c)$$

and take a complex submanifold W through (p, q) such that

$$T_{(p,q)}W = T_{(p,q)}W_c \cap J(T_{(p,q)}W_c).$$

Let $K_{(p,q)}W$ and $K_{(p,q)}V^k$ be the projective tangent cones of W and V^k ,

respectively. Since r has a maximum on V^k at (p, q) ,

$$K_{(p,q)}V^k \subseteq K_{(p,q)}W.$$

We may assume that V^k is an irreducible subvariety and then $K_{(p,q)}V^k$ has dimension $k - 1 > 0$. Therefore there exists A_0 in $K_{(p,q)}V^k$ which corresponds to a direction of $\widehat{T_{(p,q)}W_c} \cap J(T_{(p,q)}W_c)$. On the other hand since $V_{(p,q)}^k$ is a subvariety of $\widehat{M \times M}_{(p,q)}$ there exists a holomorphic curve $\varphi(t)$ in $\widehat{V}_{(p,q)}^k$ such that $\varphi(0) = A_0$ and $\varphi(t)$ locally intersects $B_{(p,q)}$ just at A_0 . The projection of $\varphi(t)$ under the quadratic transformation σ to $M \times M$ gives us a holomorphic curve $C(t)$ in V^k with $C(0) = (p, q)$, and such that if $(z^1, \dots, z^{2n})$ is a coordinate system in a neighborhood of (p, q) with $z^i(p, q) = 0$, $1 < i < 2n$, there exists an integer $d > 1$ such that

$$z^\alpha(C(t)) = z^\alpha(t) = \sum_{j=0}^{\infty} A_j^\alpha t^{d+j}, \quad 1 < \alpha < 2n;$$

here the $2n$ -tuple $(A_0^1, \dots, A_0^{2n})$ corresponds to the local homogeneous coordinates of A_0 in $\widehat{M \times M}_{(p,q)}$.

We are going to show that for every $t \in \mathbb{C}$, with $|t|$ small, there exists ζ_t with $|\zeta_t| = |t|$ and

$$r(C(\zeta_t)) - r(C(0)) > 0. \quad (9)$$

In fact, let us take the average of the function $r(C(t)) - r(C(0))$. From Theorem 3 of [2], we know that

$$\frac{1}{2\pi} \int_0^{2\pi} (r(C(te^{i\theta})) - r(C(0))) d\theta = r_{,\alpha\bar{\beta}} \left(\sum_{j=0}^{\infty} A_j^\alpha \bar{A}_j^\beta |t|^{2(d+j)} \right)$$

where $r_{,\alpha\bar{\beta}}$ stands for component of the second covariant derivative of r . On the other hand,

$$-2r_{,\alpha\bar{\beta}} A_0^\alpha \bar{A}_0^\beta = \Pi_{W_c}[A_0] + \Pi_{W_c}[JA_0]$$

which is negative because A_0 corresponds to a direction in $T_{(p,q)}W_c$. It follows that the average is positive for $|t|$ small and then (9) holds true. This contradicts the fact that r has a maximum on V^k at (p, q) . Q.E.D.

COROLLARY. *Let M be a simply connected, complete Hermitian manifold with nonpositive Riemannian sectional curvature. Then there exist no compact complex analytic subvarieties in M .*

We observe that when M is Kählerian, this corollary follows from Theorem 1 of [3]. We further note that a (non-Kählerian) Hermitian manifold satisfying the hypothesis of Theorem 3 need not be a Stein manifold, see [4].

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