ON THE SQ -UNIVERSALITY OF HNN GROUPS

BENJAMIN FINE AND MARVIN TRETKOFF

ABSTRACT. A group G is SQ -universal if every countable group can be embedded in a quotient of G . In this paper we give conditions under which HNN groups are SQ -universal. These are then applied to showing the SQ -universality of certain classes of Kleinian groups (discontinuous subgroups of $PSL_2(\mathbb{C})$). Finally some connections are mentioned between SQ -universality and the theory of ends.

Introduction. A group G is SQ -universal if every countable group can be embedded in a quotient of G . Thus all nonabelian free groups are SQ -universal [4], while it is known that all nontrivial free products except the infinite dihedral group [10], and large classes of amalgamated free products are also SQ -universal [10], [11]. Sacerdote and Schupp have extended these to certain HNN and one-relator groups [9]. In this paper we give a series of results which complement those of Sacerdote and Schupp but are somewhat less technical and more directly applicable to geometrically and arithmetically defined groups [2], [3]. This will be applied in §3 to showing the SQ -universality of a class of Kleinian groups. Finally we make some observations on the relationship between SQ -universality and the theory of ends.

1. Recall that an HNN group is a group of the form

$$G = \langle t_1, \dots, t_n, K; \text{rel } K, t_i L_i t_i^{-1} = \varphi_i(L_i) \rangle$$

where $\{L_i\}$ are subgroups of K , and $\{\varphi_i\}$ are isomorphisms $L_i \rightarrow \varphi_i(L_i) < K$. K is called the base, $\{L_i, \varphi_i(L_i)\}$ associated subgroups and $\langle t_1, \dots, t_n \rangle$ the free part. Groups of this type were used by Higman, Neumann and Neumann to show the SQ -universality of free groups of rank 2 [4]. The base K can be embedded isomorphically in G , and a subgroup structure theory for HNN groups, paralleling that for amalgamated free products, has been developed by Karrass and Solitar [5].

Now G has a free quotient of rank n if the free part has rank n , and since clearly a group with an SQ -universal quotient is itself SQ -universal we get

LEMMA. *An HNN group with free part of rank > 1 is SQ -universal.*

Received by the editors July 6, 1977.

AMS (MOS) subject classifications (1970). Primary 20E30, 20E99; Secondary 20H10.

Key words and phrases. SQ -universal, HNN group, generalized free product, Kleinian group, Fuchsian group, Klein's combination theorem, ends of a group.

© 1979 American Mathematical Society
0002-9939/79/0000-0100/\$03.00

Therefore the interesting question is in general when the free part has rank 1.

Our techniques were motivated by showing the *SQ*-universality of certain matrix groups [3], and are based in part on the following two theorems:

THEOREM I (P. M. NEUMANN) [8]. *If H is a subgroup of finite index in G , then H is *SQ*-universal if and only if G is *SQ*-universal.*

THEOREM II (M. TRETKOFF) [11]. *Let $A = A_1 * A_2$ be a nontrivial free product other than $\mathbb{Z}_2 * \mathbb{Z}_2$, and let H be a finitely generated subgroup of infinite index in A . Then the amalgamated free product of A and B identifying isomorphic subgroups $H < A$ and $M < B$ is *SQ*-universal. (We would like to note that Theorem II is a generalization of results of Schupp [10].)*

First we consider the case where the base group is *SQ*-universal. This is true for a class of matrix groups over algebraic integers [2]. The simplest conditions are when the associated subgroups are isomorphic via an inner automorphism.

THEOREM A. *Let K be *SQ*-universal, H, M conjugate subgroups of K . Then the *HNN* group*

$$G = \langle u, K; \text{rel } K, uHu^{-1} = M \rangle$$

*is *SQ*-universal.*

PROOF. Let Γ be a countable group. Then $\Gamma \subseteq K^*$, where K^* is some quotient of K .

Since H, M are conjugate their images H^*, M^* will be isomorphic in any quotient of K .

(a) Then $G^* = \langle u, K^*; \text{rel } K^*, uH^*u^{-1} = M^* \rangle$ is both a quotient of G and an *HNN* group with base K^* .

(b) Since the base can be embedded in an *HNN* group, [4] $K^* \subseteq G^*$, and then $\Gamma \subseteq K^* \subseteq G^*$, so G is *SQ*-universal.

If H, M are related via an outer automorphism, the same result can be obtained with conditions on the automorphisms.

THEOREM B. *Let K be *SQ*-universal, $H \leq K$, φ an automorphism of K such that $\varphi^2/H = 1_H$ $\{\varphi/H = \varphi$ restricted to $H\}$. Then if $M = \varphi(H)$, $G = \langle u, K; \text{rel } K, uHu^{-1} = M \rangle$ is *SQ*-universal.*

PROOF. We will construct a subgroup of finite index in a quotient of G and then show that this group is *SQ*-universal.

(a) Map G onto $G^* = G/N(u^2)$ where $N(u^2)$ is the normal closure in G of u^2 . Then $G^* = \langle u, K; \text{rel } K, uHu^{-1} = M, u^2 = 1 \rangle$.

(b) In G^* consider $G_1 = N(K)$ the normal closure of K . Then $G^*/G_1 = \langle u; u^2 = 1 \rangle$ so $|G^*: G_1| = 2$.

(c) Choosing 1, u as coset representatives and using the Reidemeister-Schreier process we get as a presentation of G_1

$$G_1 = \langle K, K_1; \text{rel } K, \text{rel } K_1, H = M_1, M = H_1 \rangle$$

where K_1 is a copy of K (conjugated by u) and H_1, M_1 are subgroups corresponding to H, M .

(d) Since $M = \varphi(H)$ and $\varphi_H^2 = 1_H$, φ interchanges H, M ($\varphi(H) = M$, $\varphi(M) = H$). Then using $\varphi: K_1 \rightarrow K$ we see that K is a homomorphic image of G_1 .

Since K is SQ-universal, so is G_1 . But G_1 has finite index in G^* , so G^* is SQ-universal by Theorem I. Finally G has G^* as a quotient, so G is SQ-universal.

The proof of Theorem B suggests the following

COROLLARY. Suppose $K = H * M$ with $H \cong M$. Then $G = \langle u, K; \text{rel } K, uHu^{-1} = M \rangle$ is SQ-universal.

PROOF. (1) If $H \cong \mathbb{Z}_2$ then K is SQ-universal. As in the proof of Theorem B, construct

$$G_1 = \langle K, K_1; \text{rel } K, \text{rel } K_1, H = M_1, M = H_1 \rangle.$$

(a) Since K_1 is a copy of K , $K_1 = H_1 * M_1$, the isomorphisms $H_1 \rightarrow M$, $M_1 \rightarrow H$ can be extended to all of K_1 . So K is a homomorphic image of G_1 , so as before G_1 is SQ-universal, and therefore G is.

(2) If $H = \mathbb{Z}_2 = \langle a; a^2 = 1 \rangle$ then G has a presentation

$$G = \langle u, a, b; a^2 = 1, b^2 = 1, uau^{-1} = b \rangle = \langle u, a, u^2 = 1 \rangle \cong \mathbb{Z} * \mathbb{Z}_2,$$

which is SQ-universal. Similar results can be proved even when the base is not SQ-universal.

THEOREM C. Suppose K is a group, H a subgroup such that $|K: N(H)| > 2$, where $N(H)$ is the normal closure of H in K . If H is conjugate to M , then $G = \langle u, K; \text{rel } K, uHu^{-1} = M \rangle$ is SQ-universal.

PROOF. Again construct G_1 , which is of finite index in a quotient of G . Further

$$G_1 = \langle K, K_1; \text{rel } K, \text{rel } K_1; H = M_1, M = H_1 \rangle,$$

where again K_1 is a copy of K and H_1, M_1 are subgroups corresponding to H, M .

(a) Consider $G_1^* = G_1/N(H)$. Since H, M are conjugate the subgroup M is in the kernel of the map $G_1 \rightarrow G_1^*$. Therefore a presentation for G_1^* is found by setting $H = 1$, so $G_1^* = \langle K, K_1; \text{rel } K, \text{rel } K_1, H = 1, H_1 = 1 \rangle$, since $M = 1$ is derivable from $H = 1$.

$$\therefore G_1^* \cong K/N(H) * K_1/N(H_1) \cong K/N(H) * K/N(H).$$

(b) Since $|K: N(H)| > 2$, neither factor is trivial or $\mathbb{Z}_2$. Therefore G_1^* is SQ-universal, and by previous arguments G is.

If we do not require H, M to be conjugate, sufficient conditions can be given by requiring the index of $N(\langle H, M \rangle)$ to be greater than 2.

THEOREM D. *If H, M are isomorphic subgroups of K and $|K: N(\langle H, M \rangle)| > 2$, $\{\langle H, M \rangle$ is the union of H and $M\}$ then $G = \langle u, K; \text{rel } K, uHu^{-1} = M \rangle$ is SQ -universal.*

PROOF. As before $G_1 = \langle K, K_1; \text{rel } K, \text{rel } K_1, H = M_1, M = H_1 \rangle$. Letting $H = 1, M = 1$ gives the quotient G_1^* of G_1 , where

$$\begin{aligned} G_1^* &= \langle K, K_1; \text{rel } K, \text{rel } K_1, H = 1, M = 1, H_1 = 1, M_1 = 1 \rangle \\ &\cong K/N(\langle H, M \rangle) * K/N(\langle H, M \rangle) \neq \mathbf{Z}_2 * \mathbf{Z}_2, \end{aligned}$$

so G is SQ -universal.

Now we give generalizations of Tretkoff's theorem (Theorem II) to HNN groups.

THEOREM E. *Suppose $A = A_1 * A_2$ is a free product other than $\mathbf{Z}_2 * \mathbf{Z}_2$. Let H be a finitely generated subgroup. If $H \cong M$ and the subgroup $\langle H, M \rangle$ generated by H, M has infinite index, then the HNN group with base A associating H, M is SQ -universal.*

PROOF. Now $G = \langle u, A; \text{rel } A, uHu^{-1} = M \rangle$.

(a) Again construct

$$G_1 = \langle A, A_1; \text{rel } A, \text{rel } A_1, H = M_1, M = H_1 \rangle.$$

(b) Since A and A_1 are identical copies, the identifications clearly yield isomorphisms, so G_1 is the amalgamated product of A and A_1 with amalgamated subgroups $\langle H, M \rangle$ and $\langle H_1, M_1 \rangle$.

(c) But A is a free product, $\langle H, M \rangle$ is finitely generated since H is, $H \cong M$ and $|A: \langle H, M \rangle| = \infty$, so G_1 is SQ -universal by Theorem II.

Finally, we get the following

THEOREM F. *Let H be a finitely generated subgroup of K . Suppose that $H \cong M$, and $\langle H, M \rangle = H * M$ (the subgroup generated H and M is their free product). Then the HNN group $G = \langle u, K; \text{rel } K, uHu^{-1} = M \rangle$ is SQ -universal.*

We note that this has applications to discontinuous groups, where, whether or not the amalgam of 2 subgroups is their free product can often be determined from the geometry.

PROOF. If $\langle H, M \rangle = K = H * M$ then the result follows from the corollary to Theorem B.

(a) So we can assume $\langle H, M \rangle = H * M \neq K$. Now let $G_1 = \langle u, H; \text{rel } H \rangle \cong \langle u \rangle * H$ and $G_2 = K$. Then G is the free product of G_1, G_2 with identifications $H = H, uHu^{-1} = M$.

(b) In G_1 the subgroup generated by H and uHu^{-1} is isomorphic to their free product. To see this consider $N(H)$ in G_1 . Using Reidemeister-Schreier,

$N(H) = * \amalg H_n$, $n = 0, \pm 1, \pm 2$, where $H_n = u^n H u^{-n}$ and $H_0 = H$, $H_1 = u H u^{-1}$.

(i) Since $\langle H, M \rangle = H * M$ and $H \cong M$, the identifications give subgroup isomorphisms and therefore G is an amalgamated free product of G_1 and G_2 .

(ii) Further $G_1 = \langle u \rangle * H$ is a free product other than $Z_2 * Z_2$ and subgroup generated by H and $u H u^{-1}$ is finitely generated since H is. Also, this subgroup has infinite index since $G_1/N(H) \cong \langle u \rangle$. Therefore G is SQ-universal by Theorem II. As immediate corollaries we get

COROLLARY. (a) Let $K = K_1 * K_2$ a free product with $H < K_1$, $M \leq K_2$, $H \cong M$, or

(b) let $K = K_1 * K_2$ an amalgamated free product, $H < K_1$, $H_1 = H_2$, $M \leq K_2$, $H \cong M$ and $H \cap H_1 = \{1\}$. Then in both cases the HNN group with base K and H, M associated is SQ-universal.

PROOF. The conditions in both cases imply $\langle H, M \rangle = H * M \neq K$.

Finally, one question which seems to suggest itself: If the base is SQ-universal, must any HNN group with that base also be SQ-universal?

2. We now give applications. A Kleinian group is a subgroup (necessarily discrete) of $PSL_2(C)$ which is discontinuous at least at one point in the extended complex plane. (Definitions and terminology will follow Maskit [7].) The most extensively studied class of Kleinian groups are Fuchsian groups, that is, discrete subgroups of $PSL_2(R)$ and their conjugates in $PSL_2(C)$. It is known [6] that a finitely generated Fuchsian group has a presentation of the form

$$\langle e_1, \dots, e_l, P_1, \dots, P_l, a_1, b_1, \dots, a_n, b_n; e_i^{n_i} = 1, \\ i = 1, \dots, l, e_1 \cdots e_l P_1 \cdots P_l [a_1, b_1] \cdots [a_n, b_n] = 1 \rangle. \quad (1)$$

P. M. Neumann [8] has shown that groups with presentation (1) are SQ-universal unless they have abelian subgroups of finite index. By a careful examination of which presentations (1) can actually be realized as Fuchsian groups we can restate this as follows,

THEOREM. A finitely generated discrete subgroup of $PSL_2(R)$ (Fuchsian group) is SQ-universal unless it is finite, cyclic or infinite dihedral.

Thus there is a base class of SQ-universal Kleinian groups. Maskit has given combination theorems, based on classical work of Klein, whereby two Kleinian groups satisfying certain algebraic and geometric conditions will again generate a Kleinian group [7]. Group theoretically these constructions lead to HNN groups or free products with amalgamations, and thus our theorems can be applied to them. Further, many of our algebraic conditions can be interpreted geometrically.

Let G_1 be a Kleinian group, H_1, H_2 subgroups of G_1 , and D_1, Δ_1, Δ_2 fundamental sets for G_1, H_1 , and H_2 respectively. Also, let G_2 be an infinite

cyclic Kleinian group generated by f . Further, assume that G_1 , H_1 , H_2 , and f satisfy all the conditions for Maskit's combination Theorem II, and let G be the Kleinian group formed. Group theoretically G is an HNN extension of G_1 with associated subgroups H_1 , H_2 and free part generated by f . Then

THEOREM 1. *If G_1 is SQ -universal, then G is SQ -universal under any of the following conditions:*

- (1) H_1 and H_2 are conjugate in G_1 .
- (2) f is in the normalizer of G_1 in $PSL_2(\mathbb{C})$ and f^2 centralizes H_1 .
- (3) $G_1 = H_1 * H_2$. Geometrically, this can be realized if $D_1 = \Delta_1 \cap \Delta_2$ and H_1 , D_1 , H_2 , D_2 satisfy conditions for Klein's combination theorem for free products [6].
- (4) G_1 is a free product (not infinite dihedral). H_1 is finitely generated and $\langle H_1, H_2 \rangle$ has infinite index. In particular, this holds if $\text{rank } G_1 \geq 3$ and H_1 is cyclic.
- (5) $G_1 = \Gamma_1 * \Gamma_2$ with $H_1 < \Gamma_1$, $H_2 \leq \Gamma_2$.

Thus, starting with a Fuchsian group, which is the usual procedure, we will in many cases again generate SQ -universal Kleinian groups.

THEOREM 2. *G is SQ -universal if H_1 is finitely generated and $\langle H_1, H_2 \rangle = H_1 * H_2$. In particular this holds if H_1 , H_2 satisfy the conditions of Klein's combination theorem.*

We will note that our results dealing with the normal closures of the associated subgroups can also be applied but in general these are very difficult to determine in the geometric setting. Further, cyclic associated subgroups as in (4) above were the first cases handled by Maskit.

Tretkoff's theorem [11] can be used directly for groups formed by combination Theorem I, which become free products with amalgamations. A particularly nice case is

THEOREM 3. *If G_1 is a free product (not infinite dihedral) and H_1 is cyclic, then any Kleinian group G formed from G_1 , H_1 by combination Theorem I is SQ -universal.*

Cyclic amalgamations were the first cases handled by Maskit.

Groups formed by the two combination theorems applied a finite number of times, starting with a Fuchsian group, are termed *constructible*. Thus a large subclass of the constructible groups are SQ -universal.

3. The number of ends of a group G is given by $1 + \dim H^1(G, \mathbb{Z}_2 G)$ [1], and is directly related to the topological theory of ends via the graph of the group. A structure theory was developed by Stallings for finitely generated groups with more than one end. An excellent exposition of this theory is given in [5]. Many of the conditions were similar to those for SQ -universality so some connections will be presented.

It is known that nonlocally finite groups have 1, 2 or ∞ ends. Since quotients and subgroups of locally finite groups are locally finite, SQ -universal groups cannot be locally finite; therefore SQ -universal groups must have 1, 2 or ∞ ends. However, we can say more. We will assume the groups are finitely generated.

REMARK 1. An SQ -universal group has 1 or ∞ ends. It is known [1] that if G has 2 ends, it has an infinite cyclic subgroup of finite index and so cannot be SQ -universal.

We now show that there exist SQ -universal groups with both 1 and ∞ ends.

REMARK 2. If G is torsion free with an infinite number of ends, G is SQ -universal.

Again from [1] G must be a nontrivial free product other than $\mathbb{Z}_2 * \mathbb{Z}_2$, and therefore SQ -universal. Since groups with an infinite number of ends must decompose as either generalized free products or HNN groups, it follows that if an SQ -universal group is indecomposable with respect to generalized free products or HNN groups it must have one end.

LEMMA. *There exist SQ -universal groups with one end.*

PROOF. The group $PSL_2(\mathbb{O}_3)$ where $\mathbb{O}_3$ is the ring of integers in $Q(\sqrt{-3})$ is SQ -universal [3], but indecomposable with respect to generalized free products or HNN groups [2]. A second example is the fundamental group of any Riemann surface of finite genus ≥ 2 . These are SQ -universal [8], but torsion-free and not free products.

Since groups with an infinite number of ends are all products, we ask: Is every finitely generated group with an infinite number of ends SQ -universal?

ACKNOWLEDGEMENT. We would like to thank the referee and editors for their helpful comments.

REFERENCES

1. D. E. Cohen, *Groups of cohomological dimension one*, Lecture Notes in Math., vol. 245, Springer-Verlag, Berlin and New York, 1972.
2. B. Fine, *The HNN and generalized free product structure of certain linear groups*, Bull. Amer. Math. Soc. **81** (1976), 413–416.
3. B. Fine and M. Tretkoff, *The SQ -universality of certain arithmetically defined linear groups*, J. London Math. Soc. (2) **13** (1976), 65–68.
4. G. Higman, B. H. Neumann and H. Neumann, *Embedding theorems for groups*, J. London Math. Soc. **24** (1949), 247–254.
5. A. Karrass and D. Solitar, *Subgroups of HNN groups and groups with one defining relation*, Canad. J. Math. **23** (1971), 627–643.
6. J. Lehner, *Discontinuous groups and automorphic functions*, Math. Surveys, No. 8, Amer. Math. Soc., Providence, R. I., 1964.
7. B. Maskit, *On Klein's combination theorem*. III, Advances in the Theory of Riemann Surfaces, Ann. of Math. Studies, No. 66, Princeton Univ. Press, Princeton, N. J., 1971, pp. 297–316.
8. P. M. Neumann, *The SQ -universality of some finitely presented groups*, J. Austral. Math. Soc. **16** (1973), 1–6.

9. G. Sacerdote and P. Schupp, *SQ-universality in HNN groups and one-relator groups*, J. London Math. Soc. (2) 7 (1974), 733–740.

10. P. Schupp, *Small cancellation theory over free products with amalgamation*, Math. Ann. 193 (1971), 255–264.

11. M. Tretkoff, *Covering space proofs in combinatorial group theory*, Comm. Algebra 3 (1975), 429–457.

DEPARTMENT OF MATHEMATICS, FAIRFIELD UNIVERSITY, FAIRFIELD, CONNECTICUT 06430

DEPARTMENT OF MATHEMATICS, STEVENS INSTITUTE OF TECHNOLOGY, HOBOKEN, NEW JERSEY 07030