AN L^1 REMAINDER THEOREM FOR AN INTEGRODIFFERENTIAL EQUATION WITH ASYMPTOTICALLY PERIODIC SOLUTION

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ABSTRACT. For a certain integrodifferential equation of Volterra type on $(0, \infty)$, with piecewise linear convolution kernel, it is shown that the solution is $u(t) = \alpha \cos \beta t + \rho(t)$, with $\rho \in L^1(0, \infty)$ and α and β constant; u' is represented similarly.

1. Introduction. Let $c \geq 0$, and suppose

$$\begin{aligned} a(t) \text{ is nonnegative, nonincreasing, and convex on } (0, \infty) \text{ with} \\ a \in L^1(0, 1) \text{ and } a(\infty) = 0. \end{aligned} \quad (1.1)$$

It was proved in [3] that if

$$u'(t) + \int_0^t [c + a(t-s)]u(s) ds = 0 \quad (t > 0), u(0) = 1, \quad (1.2)$$

($' = d/dt$) then $u(t) \rightarrow 0$ ($t \rightarrow \infty$), except in the special case where a has the piecewise linear form

$$a(t) = \sum_{k=1}^{\infty} \delta_k \left(1 - \frac{\min\{t, kt_0\}}{kt_0} \right), \quad (1.3)$$

with $t_0 > 0$ and

$$\delta_k \geq 0, \quad 0 < a(0) = \sum_{k=1}^{\infty} \delta_k \equiv \delta < \infty, \quad (1.4)$$

$$\omega \equiv \sqrt{\delta + c} = 2\pi j/t_0 \quad \text{for some integer } j. \quad (1.5)$$

If (1.3), (1.4), and (1.5) hold, we have

$$u_1(t) \equiv u(t) - \frac{2}{\gamma} \cos \omega t \rightarrow 0 \quad (t \rightarrow \infty) \quad (1.6)$$

with $\gamma = (3\delta + 2c)/(\delta + c)$. Results of S. I. Grossman and R. K. Miller [2] (for $a \in L^1(0, \infty)$) and of D. F. Shea and S. Wainger [7] show that $u, u' \in L^1(0, \infty)$ if a has no representation (1.3) satisfying (1.4) and (1.5). Here we establish that

$$\int_0^{\infty} |u_1(t)| + |u'_1(t)| dt < \infty \quad (1.7)$$

Received by the editors May 1, 1978 and, in revised form, June 26, 1978.

AMS (MOS) subject classifications (1970). Primary 45D05, 45J05, 45M05.

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0002-9939/79/0000-0108/\$02.75

in the asymptotically periodic case if

$$\int_0^\infty a(t) dt < \infty, \quad (1.8)$$

and we demonstrate that (1.7) need not hold if (1.8) fails to hold.

THEOREM 1.1. *Let $c \geq 0$ and let (1.3), (1.4), (1.5), and (1.8) hold. Let u be the solution of (1.2), and define u_1 as in (1.6). Then u_1 satisfies (1.7).*

Miller [6] and Jordan and Wheeler [5] give remainder estimates like (1.7) for various classes of integral and integrodifferential equations. We use special properties of kernels satisfying (1.1) to avoid moment conditions assumed for analogous problems in [5], [6]. In particular, (1.1) and (1.8) imply that the Fourier transform

$$\hat{a}(\tau) = \int_0^\infty a(t) e^{-i\tau t} dt$$

is twice continuously differentiable.

The key to the proof is an estimate for the difference between $\hat{a}(\tau)$ and its linear approximation at $\tau = \pm\omega$. Since this result may have further applications, we prove it in a slightly generalized form in Lemma 3.1.

The author thanks R. L. Wheeler for a number of corrections.

2. Preliminaries. In this section we summarize some results from [3], valid under the assumptions of Theorem 1.1. Here and below we assume that $t_0 = 1$; this may be accomplished by rescaling.

The Laplace transform

$$\begin{aligned} a^*(z) &= \int_0^\infty e^{-zt} a(t) dt \\ &= \frac{\delta}{z} + \frac{1}{z^2} \sum_{k=1}^\infty \frac{\delta_k}{k} (e^{-kz} - 1) \end{aligned} \quad (2.1)$$

is analytic for $\operatorname{Re} z > 0$ and continuous for $\operatorname{Re} z \geq 0$. Moreover, $u_1(t) \rightarrow 0$ ($t \rightarrow \infty$), and

$$u_1^*(z) = \frac{1}{z + c/z + a^*(z)} - \frac{2z}{\gamma(z^2 + \omega^2)} \quad (\operatorname{Re} z > 0). \quad (2.2)$$

Note that

$$u_1^*(\bar{z}) = \bar{u}_1^*(z) \quad (2.3)$$

($\bar{}$ = complex conjugate). u_1^* has a continuous extension to $\{\operatorname{Re} z \geq 0, z \neq \pm i\omega\}$. A little rearranging shows that

$$\begin{aligned} z + \frac{c}{z} + a^*(z) &= \gamma(z - i\omega) - \left(\frac{1}{z} + \frac{\delta(z + i\omega)}{z^2 \omega^2} \right) (z - i\omega)^2 \\ &\quad + z^{-2} \sum_{k=1}^\infty \frac{\delta_k}{k} [e^{-k(z - i\omega)} - 1 + k(z - i\omega)]. \end{aligned} \quad (2.4)$$

By considering the cases where $|k(z - i\omega)| \geq 1$, < 1 , one sees that

$$\left| \frac{e^{-k(z-i\omega)} - 1 + k(z-i\omega)}{k(z-i\omega)^2} \right| \leq 3k \quad (1 \leq k < \infty, \operatorname{Re} z \geq 0, z \neq i\omega).$$

Since $\frac{1}{2} \sum k \delta_k = \int_0^\infty a(t) dt < \infty$, we may use dominated convergence to see that

$$\lim_{\substack{z \rightarrow i\omega \\ \operatorname{Re} z > 0}} (z - i\omega)^{-2} \sum_{k=1}^{\infty} \frac{\delta_k}{k} [e^{-k(z-i\omega)} - 1 + k(z-i\omega)] = \int_0^\infty a(t) dt.$$

By (2.2), (2.3), and (2.4), u_1^* is continuous and bounded in $\{\operatorname{Re} z \geq 0\}$.

Let

$$f(\tau) = u_1^*(i\tau) \quad (-\infty < \tau < \infty). \quad (2.5)$$

We shall show that f is absolutely continuous with

$$\int_{-\infty}^{\infty} |f'(\tau)| d\tau < \infty. \quad (2.6)$$

To complete the proof that $u_1 \in L^1$, we adapt an argument from [7], based on the estimate

$$\int_0^\infty \frac{|F^\vee(t)|}{t} dt \leq \frac{1}{2} \int_{-\infty}^\infty |F(\tau)| d\tau$$

for functions F of Hardy class H^1 in the lower half-plane. ($F = f'$ in our case; $^\vee$ denotes the inverse Fourier transform.) The proof for u_1' involves a similar argument.

3. A linear approximation. In this section we consider kernels of the general type (1.1).

LEMMA 3.1. *Let $a(t)$ satisfy (1.1) and (1.8). Let $p(\tau) = q(\tau) + \hat{a}(\tau)$ on an interval $\{|\tau - \nu| \leq \varepsilon\}$ with $\varepsilon < |\nu|$, where $q \in C^2$ and q'' is Lipschitz continuous on $[\nu - \varepsilon, \nu + \varepsilon]$. Assume that $p(\nu) = 0$, $p'(\nu) \neq 0$, and $p(\tau) \neq 0$ ($0 < |\tau - \nu| \leq \varepsilon$). Let $P(\nu) = p''(\nu)/2[p'(\nu)]^2$,*

$$P(\tau) = [p'(\nu)(\tau - \nu)]^{-1} - [p(\tau)]^{-1} \quad (\tau \neq \nu). \quad (3.1)$$

Then P is absolutely continuous on $\{|\tau - \nu| \leq \varepsilon\}$.

PROOF. As is shown in [1, Proof of Lemma 4.1], under the conditions of the lemma we have

$$\hat{a}(\tau) = \tau^{-2} \int_0^\infty (1 - i\tau t - e^{-i\tau t}) da'(t) \quad (\tau > 0). \quad (3.2)$$

(By symmetry, we may and shall assume that $\nu > 0$.) Here a' is redefined where necessary so as to be continuous from the left on $(0, \infty)$. Integration by

parts, together with (1.1) and (1.8), shows that

$$\int_0^\infty t^2 da'(t) < \infty. \quad (3.3)$$

Therefore $\hat{a} \in C^2(0, \infty)$, and it follows easily that P is continuous for $|\tau - \nu| < \varepsilon$ and twice continuously differentiable for $0 < |\tau - \nu| < \varepsilon$. Thus we need only show that P' is of class L^1 on the interval.

We have

$$P'(\tau) = \frac{p'(\nu)p'(\tau) - (\tau - \nu)^{-2}p^2(\tau)}{p'(\nu)p^2(\tau)} \quad (\tau \neq \nu). \quad (3.4)$$

But $(\tau - \nu)^{-1}(p'(\nu) - p(\tau)(\tau - \nu)^{-1}) \rightarrow -\frac{1}{2}p''(\nu) (\tau \rightarrow \nu)$, so

$$\begin{aligned} p'(\nu)p'(\tau) - (\tau - \nu)^{-2}p^2(\tau) \\ = p'(\nu) \left[p'(\tau) + p'(\nu) - \frac{2p(\tau)}{\tau - \nu} \right] - \left[p'(\nu) - \frac{p(\tau)}{\tau - \nu} \right]^2 \\ = p'(\nu) \left[p'(\tau) + p'(\nu) - \frac{2p(\tau)}{\tau - \nu} \right] + O(\tau - \nu)^2 \quad (\tau \rightarrow \nu). \end{aligned} \quad (3.5)$$

By (3.2), (3.3), and the hypothesis on q ,

$$p(\tau) = \nu^{-2}b(\tau) + h(\tau), \quad (3.6)$$

where $h \in C^2[\nu - \varepsilon, \nu + \varepsilon]$, $h''(\tau) - h''(\nu) = O(\tau - \nu) (\tau \rightarrow \nu)$, and

$$b(\tau) = \int_0^\infty (1 - i\tau t - e^{-i\tau t}) da'(t).$$

Then by Taylor's Theorem, there exist τ_1, τ_2 between τ and ν such that

$$\begin{aligned} h'(\tau) + h'(\nu) - 2 \frac{h(\tau) - h(\nu)}{\tau - \nu} &= h'(\tau) - h'(\nu) - h''(\tau_1)(\tau - \nu) \\ &= [h''(\tau_2) - h''(\tau_1)](\tau - \nu) \\ &= O(\tau - \nu)^2 \quad (\tau \rightarrow \nu). \end{aligned} \quad (3.7)$$

Since $b'(\tau) = \int_0^\infty it(e^{-i\tau t} - 1)da'(t)$,

$$\begin{aligned} &|b'(\tau) + b'(\nu) - 2(\tau - \nu)^{-1}(b(\tau) - b(\nu))| \\ &\leq \int_0^\infty \left| i[e^{-i\tau t} + e^{-i\nu t}] + 2 \frac{e^{-i\tau t} - e^{-i\nu t}}{t(\tau - \nu)} \right| t da'(t). \end{aligned}$$

The function $J(\tau, t)$ inside the absolute value signs in this integral can be written $J(\tau, t) = ie^{-i\nu t}K((\tau - \nu)t)$, where $K(x)$ is a C^2 function with $K(0) = K'(0) = 0$, $K''(0) = -\frac{1}{3}$, and $|K(x)| \leq M < \infty$ for all x . We choose $\mu > 0$ such that $|K(x)| \leq x^2$ if $|x| < \mu$. Then by Fubini's Theorem and (3.3) (with the change of variable $\sigma = \pm(\tau - \nu)$, and with $E_\mu = E_\mu(\tau) = \{t | \tau - \nu| <$

$\mu\}$, $E'_\mu = \{\mu < t|\tau - \nu| < \infty\}$,

$$\begin{aligned} & \int_{\nu-\varepsilon}^{\nu+\varepsilon} (\tau - \nu)^{-2} |b(\tau) + b(\nu) - 2(\tau - \nu)^{-1}(b(\tau) - b(\nu))| d\tau \\ & \leq \int_{\nu-\varepsilon}^{\nu+\varepsilon} (\tau - \nu)^{-2} \left[\int_{E'_\mu} (\tau - \nu)^2 t^3 da'(t) + M \int_{E'_\mu} t da'(t) \right] d\tau \\ & \leq 2 \int_0^\infty \left[t^3 \int_0^{\mu/t} d\sigma + tM \int_{\mu/t}^\infty \sigma^{-2} d\sigma \right] da'(t) \\ & = 2(\mu + M\mu^{-1}) \int_0^\infty t^2 da'(t) < \infty. \end{aligned}$$

From (3.5), (3.6) and (3.7) (recall that $p(\nu) = 0$) we see, therefore, that

$$\int_{\nu-\varepsilon}^{\nu+\varepsilon} |p'(\nu)p'(\tau) - (\tau - \nu)^{-2}p^2(\tau)|(\tau - \nu)^{-2} d\tau < \infty,$$

so $P' \in L^1(\nu - \varepsilon, \nu + \varepsilon)$, and our assertion follows.

4. Proof of Theorem 1.1. From the proof of Lemma 3.1, we see that $\hat{a}(\tau) \in C^2$ for $\tau \neq 0$. In [7, Lemma 1] it is shown (from (3.2)) under weaker hypotheses than ours that

$$|\hat{a}'(\tau)| \leq 40 \int_0^{1/|\tau|} ta(t) dt \leq 40a(0)/|\tau|^2 \quad (\tau \neq 0). \quad (4.1)$$

Then by (1.8),

$$\begin{aligned} \int_{-1}^1 |\hat{a}'(\tau)| d\tau & \leq 80 \int_1^\infty y^{-2} \int_0^y ta(t) dt dy \\ & = 80 \int_0^\infty ta(t) \int_{\max\{1,t\}}^\infty y^{-2} dy dt < \infty. \end{aligned} \quad (4.2)$$

From (2.5), (2.2), (4.1), (4.2) and the continuity of $f(\tau)$ it follows that if $0 < \varepsilon < \omega$,

$$\int_{\{|\tau|-\omega>\varepsilon\}} |f'(\tau)| d\tau < \infty. \quad (4.3)$$

Set $p(\tau) = i\tau + c(i\tau)^{-1} + \hat{a}(\tau)$, and note (from (2.4)) that

$$p(\omega) = 0, \quad p'(\omega) = i\gamma. \quad (4.4)$$

Then

$$f(\tau) = \frac{1}{p(\tau)} - \frac{1}{p'(\omega)(\tau - \omega)} + r(\tau) \quad (0 < |\tau - \omega| \leq \varepsilon), \quad (4.5)$$

where $r \in C^1[\omega - \varepsilon, \omega + \varepsilon]$. Thus Lemma 3.1 (with $\nu = \omega$), the fact that $p(-\tau) = \bar{p}(\tau)$, and (4.3) imply that f is absolutely continuous and satisfies (2.6). By [4, Lemma 3.1] (which merely summarizes a procedure used in [7]), there is a function $w \in L^1(0, \infty)$ such that $u_1^*(z) = w^*(z)$ ($\operatorname{Re} z > 0$). By uniqueness of Laplace transforms, $u_1 \in L^1(0, \infty)$.

A similar argument applies to u'_1 . By (1.2) and (1.8), u'_1 is bounded and

$$\begin{aligned}(u'_1)^*(z) &= \frac{-a^*(z) - c/z}{z + c/z + a^*(z)} + \frac{2\omega^2}{\gamma(z^2 + \omega^2)} \\ &= \left(\frac{2}{\gamma} - 1\right) + zu_1^*(z) \quad (\operatorname{Re} z > 0).\end{aligned}$$

Thus $(u'_1)^*(z)$ is analytic for $\operatorname{Re} z > 0$ and continuous and bounded for $\operatorname{Re} z > 0$ with

$$\begin{aligned}(u'_1)^*(i\tau) &\equiv g(\tau) = \frac{-\hat{a}(\tau) + ic/\tau}{p(\tau)} + \frac{2\omega^2}{\gamma(\omega^2 - \tau^2)} \\ &= O(\tau^{-1}) \quad (|\tau| \rightarrow \infty).\end{aligned}\quad (4.6)$$

Then

$$g'(\tau) = \frac{i\hat{a}'(\tau) - i\tau\hat{a}''(\tau) + 2c/\tau}{p^2(\tau)} + \frac{4\omega^2\tau}{\gamma(\omega^2 - \tau^2)^2}.\quad (4.7)$$

From (4.7), (4.1), and (4.2) it is clear that with $0 < \varepsilon < \omega$ we have

$$\int_{\{|\tau| - \omega > \varepsilon\}} |g'(\tau)| d\tau < \infty.\quad (4.8)$$

Near $\tau = \omega$, (4.4) and (4.6) show that

$$g(\tau) = [-\hat{a}(\tau) + ic/\tau] \left[\frac{1}{p(\tau)} - \frac{1}{p'(\omega)(\tau - \omega)} \right] - \frac{\hat{a}(\tau) - \hat{a}(\omega)}{i\gamma(\tau - \omega)} + R(\tau),$$

where $R(\tau) \in C^2[\omega - \varepsilon, \omega + \varepsilon]$. Since $\hat{a} \in C^2$, an application of Lemma 3.1, together with (2.3) and (4.8) gives $\int_{-\infty}^{\infty} |g'(\tau)| d\tau < \infty$, and we finish as before. This proves (1.7).

5. An example. In Theorem 1.1 our assumption that $a \in L^1(0, \infty)$ cannot be dropped. For example, let $c = 0$, $\delta_k = k^{-3/2}$. If u_1 were in $L^1(0, \infty)$, the function $[p(\tau)]^{-1} - [i\gamma(\tau - \omega)]^{-1}$ would be continuous at $\tau = \omega$ (see (4.5)); then

$$p(\tau) - i\gamma(\tau - \omega) = O(\tau - \omega)^2 \quad (\tau \rightarrow \omega).\quad (5.1)$$

But by the first part of (2.4) (valid here), there is an $m < \infty$ with

$$\begin{aligned}&\left| \sum_{k=1}^{\infty} k^{-5/2} (\cos k(\tau - \omega) - 1) \right| \\ &< m |p(\tau) - i\gamma(\tau - \omega)| + O(\tau - \omega)^2 \quad (\tau \rightarrow \omega).\end{aligned}$$

Each term in the sum is negative, so if we let $(\tau - \omega) = \pi/2n$ and neglect all

terms with $-\pi/2 < k(\tau - \omega) < \pi/2 \pmod{2\pi}$, we see that $|\sum_{k=n}^{\infty} \dots|$ is at least

$$\frac{1}{2} \sum_{k=n}^{\infty} k^{-5/2} \geq \frac{1}{3} n^{-3/2} = \text{const } (\tau - \omega)^{3/2};$$

this contradicts (5.1).

REFERENCES

1. R. W. Carr and K. B. Hannsgen, *A nonhomogeneous integrodifferential equation in Hilbert space*, SIAM J. Math. Anal. (to appear).
2. S. I. Grossman and R. K. Miller, *Nonlinear Volterra integrodifferential systems with L^1 -kernels*, J. Differential Equations **13** (1973), 551–566.
3. K. B. Hannsgen, *Indirect abelian theorems and a linear Volterra equation*, Trans. Amer. Math. Soc. **142** (1969), 539–555.
4. ———, *Uniform boundedness in a class of Volterra equations*, SIAM J. Math. Anal. **6** (1975), 689–697.
5. G. S. Jordan and R. L. Wheeler, *Structure of resolvents of Volterra integral and integrodifferential systems* (to appear).
6. R. K. Miller, *Structure of solutions of unstable linear Volterra integrodifferential equations*, J. Differential Equations **15** (1974), 129–157.
7. D. F. Shea and S. Wainger, *Variants of the Wiener-Lévy theorem, with applications to stability problems for some Volterra integral equations*, Amer. J. Math. **97** (1975), 312–343.

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