THE STABLE SOLUTIONS OF QUADRATIC MATRIX EQUATIONS

STEPHEN CAMPBELL AND JOHN DAUGHTRY

ABSTRACT. The authors determine which solutions K to the quadratic matrix equation $XBX + XA - DX - C = 0$ are "stable" in the sense that all small changes in the coefficients of the equation produce equations some of whose solutions are close to K (in the metric determined by the operator norm). Our main result is that a solution is stable if and only if it is an isolated solution. (The isolated solutions already have a simple characterization in terms of the coefficient matrices.) It follows that each equation has only finitely many stable solutions.

Equivalently, we identify the stable invariant subspaces for an operator T on a finite-dimensional space as the isolated invariant subspaces.

Let H and K be finite-dimensional complex inner product spaces with A in $B(H)$, D in $B(K)$, B in $B(K, H)$, and C in $B(H, K)$, where $B(\cdot)$ denotes a set of bounded linear transformations in the usual manner. We study the operator equation

$$F(X) = XBX + XA - DX - C = 0 \quad (1)$$

for X in $B(H, K)$. The study of (1) is important for many applications, including the quadratic eigenvalue problem and aspects of the theory of optimal control. (See [5], [1], and the bibliographies there.)

In general, solutions to (1) may fluctuate wildly when the coefficients are changed slightly. Indeed, the equation $XA - AX = 0$ (for any square matrix A) has many solutions while $XA - AX = \epsilon I$ has none (where ϵ is any nonzero complex number and I is the identity matrix of appropriate dimension). Obviously, this possibility is devastating when the choice of coefficient matrices may involve small errors.

It is well known that if K is a solution to (1) such that $F'(K)$ (the Fréchet derivative of F at K) is invertible, then K is a "stable" solution in the following sense: for any $\epsilon > 0$, making each coefficient of the perturbed equation $X\mathcal{B}X + X\mathcal{A} - \mathcal{D}X - \mathcal{C} = 0$ sufficiently close (in operator norm) to the corresponding coefficient of (1) ensures that the new equation has a solution $\mathcal{K}$ with $\|\mathcal{K} - K\| < \epsilon$. ($\|\cdot\|$ denotes the operator norm.) Of course, the invertibility of $F'(K)$ is equivalent to the disjointness of the spectra of $A + BK$ and $D - KB$ [8]. Our necessary and sufficient condition for stability is closely related to this criterion:

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THEOREM. *The following properties of a solution K to (1) are equivalent:*

- (i) K is a stable solution.
- (ii) O is a stable solution to $XBX + X(A + BK) - (D - KB)X = 0$.
- (iii) The common eigenvalues for $A + BK$ and $D - KB$, considered as eigenvalues of

$$T = \begin{pmatrix} A + BK & B \\ 0 & D - KB \end{pmatrix},$$

acting on $\mathbf{H} \oplus \mathbf{K}$, have geometric multiplicity one.

- (iv) K is an isolated point of the solution set.
- (v) Every noncyclic generalized eigenspace $\mathbf{M}$ for T (defined in (iii)) satisfies $\mathbf{M} \cap \mathbf{H} = \{0\}$ or $\mathbf{M}$.
- (vi) The intersection of the graph of K with each noncyclic generalized eigenspace $\mathbf{M}$ for $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is either $\{0\}$ or $\mathbf{M}$.

REMARK. It follows from (vi) that (1) can have only finitely many stable solutions.

The equivalence of (iii)–(vi) was proved in [2] and will be assumed here. (The equivalence of (iv) and (vi) is really a translation to a different setting of a result from [4].) What we shall prove here is the equivalence of (i) and (vi) and the equivalence of (ii) and (v). We will need the homeomorphism g between solutions to (1) and certain invariant subspaces for S . Define the distance between two subspaces of $\mathbf{H} \oplus \mathbf{K}$ to be the (operator) distance between their orthogonal projections. Then g takes K (a solution to (1)) to the graph of K . The range of g is the set of invariant subspaces for S whose distance from $\mathbf{H}$ is less than one [2, Theorem 1]. We call an invariant subspace $\mathbf{M}$ for an operator S “stable” if given $\epsilon > 0$ there exists $\delta > 0$ such that $\|S - L\| < \delta$ for some operator L implies that L has an invariant subspace $\mathbf{M}'$ with $\text{dist}(\mathbf{M}, \mathbf{M}') < \epsilon$.

In order to establish the desired equivalences, it suffices to prove:

THEOREM. *An invariant subspace for an operator S on a finite-dimensional Hilbert space $\mathbf{L}$ is stable if and only if it is an isolated point of the set of invariant subspaces for S .*

PROOF. Suppose $\mathbf{M}$ is an isolated invariant subspace for S , but there exist an $\epsilon > 0$ and a sequence S_m of operators converging to S such that $\text{dist}(\mathbf{N}_m, \mathbf{M}) > \epsilon$ for each choice of $\mathbf{N}_m$, an invariant subspace for S_m . Let $\mathbf{L} = \bigoplus_{i=1}^n \mathbf{M}_i$ be the primary decomposition for S . (Each $\mathbf{M}_i$ is the span of all the generalized eigenvectors that go with a particular eigenvalue of S .) The primary summands for S_m may be partitioned so that if $\mathbf{L}_{i,m}$ is the span of one equivalence class of summands, then $\mathbf{L}_{i,m}$ converges to $\mathbf{M}_i$ as m approaches infinity [7, p. 106]. There exists some index (which we call 1) such that $\mathbf{M} \cap \mathbf{M}_1$ is not the limit of any sequence $\mathbf{Q}_m$ of subspaces of $\mathbf{L}_{1,m}$ with $\mathbf{Q}_m$ invariant for S_m . We may choose a sequence $\mathbf{Q}_m \subset \mathbf{L}_{1,m}$ of invariant

subspaces for S_m such that

$$\dim Q_m = \dim(M \cap M_1)$$

and Q_m converges to $Q \subset M$. Q is then an invariant subspace for S . However, M_1 is cyclic by the Douglas-Pearcy characterization of isolation (vi). Therefore, $M \cap M_1$ is the unique invariant subspace for the restriction of S to M_1 having its particular dimension. Consequently, $Q = M \cap M_1$, which gives us a contradiction. Thus isolated solutions are stable.

Now suppose that M is a nonisolated invariant subspace for S . Suppose that M_1 is one of the noncyclic primary summands for S such that $M \cap M_1$ is a proper subspace of M_1 . Let S_1 be the restriction of S to M_1 . We shall demonstrate that $M \cap M_1$ is unstable for S_1 . Let B be a basis for M_1 with respect to which S_1 is in Jordan form (with the off-diagonal ones *below* the main diagonal). $B = \cup B_j$ where each B_j is a basis for one of the cyclic summands in M_1 (i.e., each B_j corresponds to one Jordan cell). We claim that the dimension of the intersection of M with the span of B_j is equal to the number of elements in $M \cap B_j$ for all j . To see this, let $M_{1,j}$ denote the span of B_j . We may assume that $M \cap M_1$ is stable, since otherwise we have finished this part of the proof. Choose distinct complex numbers ε_j near 0 and define T on M_1 by

$$Tx = S_1x + \varepsilon_jx$$

for x in $M_{1,j}$. Then each invariant subspace for T splits into a direct sum of its intersections with the subspaces $M_{1,j}$, and the matrix of T with respect to B_j is a Jordan cell. By revising our choice of the numbers ε_j , we may take T arbitrarily close to S_1 . By the stability of M , M is a limit of some sequence of invariant subspaces for the operators T obtained by varying the numbers ε_j , but the invariant subspaces of T do not change with the choice of these numbers, and each invariant subspace for T satisfies the condition desired for M . Therefore M also splits.

In particular, there is some j such that $B_j \cap M$ contains a nonzero vector y . Renumbering the sets B_j if necessary, y belongs to $B_1 \cap M$. Choose some nonzero number ε , and consider the operator W_ε on M_1 obtained from S_1 by replacing each zero entry below the main diagonal in the matrix for S_1 with ε . The vectors $W_\varepsilon^k y$ ($k \geq 1$) span a space including the span of B_j ($j > 1$). Because W_ε approaches S_1 as ε approaches 0 and $M \cap M_1$ is stable, we may conclude that the distance from y to some invariant subspace M_ε for W_ε approaches 0. Yet if $B = \{y_1, y_2, \dots, y_n\}$, any invariant subspace for W_ε is simply the span of a subset of B having the form $\{y_i, y_{i+1}, \dots, y_n\}$. It follows that y belongs to M_ε for sufficiently small ε , where $\{M_\varepsilon\}$ is a collection of subspaces approaching $M \cap M_1$. Thus the span of B_j is contained in M_ε for $j > 1$ and sufficiently small ε . It follows that $B_j \subset M \cap M_1$ for $j > 1$. (j assumes the value 2 because M_1 is not cyclic for S_1 .) Knowing that $B_2 \subset M \cap M_1$, we may reproduce the preceding argument with B_1 and B_2 interchanged,

and we may take $y = y_1$, the first element of $\mathbf{B}$. This time we conclude that all of $\mathbf{B}$ is contained in $\mathbf{M}$, contrary to the fact that $\mathbf{M} \cap \mathbf{M}_1$ was a proper subspace of $\mathbf{M}_1$. This error must be the result of the supposition that $\mathbf{M} \cap \mathbf{M}_1$ was stable for S_1 , so we conclude that $\mathbf{M} \cap \mathbf{M}_1$ is unstable for S_1 .

To prove that $\mathbf{M}$ is unstable for S , define S_ε by $S_\varepsilon = S$ on $\mathbf{M}_k$ for $k \neq 1$ and choose the restriction of S_ε to $\mathbf{M}_1$ so that $S_\varepsilon \rightarrow S_1$ as $\varepsilon \rightarrow 0$ and the invariant subspaces of S_ε in $\mathbf{M}_1$ are bounded away from $\mathbf{M} \cap \mathbf{M}_1$. Every invariant subspace $\mathbf{M}_\varepsilon$ for S_ε splits $\mathbf{M}_\varepsilon = \bigoplus_i (\mathbf{M}_\varepsilon \cap \mathbf{M}_i)$, so the fact that $\mathbf{M}_\varepsilon \cap \mathbf{M}_1$ is bounded away from $\mathbf{M} \cap \mathbf{M}_1$ implies that $\mathbf{M}_\varepsilon$ is bounded away from $\mathbf{M}$. This proves that $\mathbf{M}$ is unstable for S , completing the proof of the theorem. $\square$

The argument that isolation implies stability given in the preceding theorem is easily modified to show that if S is a linear operator on a finite-dimensional space and S' is sufficiently near S , then every connected component $\mathbf{C}'$ of invariant subspaces for S' is close to some connected component $\mathbf{C}$ of invariant subspaces for S in the sense that

$$\sup_{\mathbf{M}' \in \mathbf{C}'} \inf_{\mathbf{M} \in \mathbf{C}} \text{dist}(\mathbf{M}', \mathbf{M})$$

goes to 0 as S' approaches S .

We begin the proof with the characterization of the connected components of invariant subspaces for S obtained by Douglas and Percy in [4]: two invariant subspaces are in the same component if and only if their intersections with each generalized eigenspace for S have the same dimension. Now suppose that there exist a sequence $\{S_m\}$ of operators converging to S and a sequence $\{\mathbf{C}_m\}$ with each $\mathbf{C}_m$ a connected component of invariant subspaces for S_m satisfying

$$\sup_{\mathbf{M}_m \in \mathbf{C}_m} \inf_{\mathbf{M} \in \mathbf{C}} \text{dist}(\mathbf{M}_m, \mathbf{M}) > \delta > 0$$

for every component $\mathbf{C}$ of invariant subspaces for S . Define $\mathbf{L}_{i,m}$ (as in the proof of the preceding theorem) to be the span of certain primary summands for S_m so that $\mathbf{L}_{i,m} \rightarrow \mathbf{M}_i$ where $\bigoplus_i \mathbf{M}_i$ is the primary decomposition for S . By the same arguments as before we may assume that for any $\mathbf{C}$ as above there exists $\mathbf{M}_m$ in $\mathbf{C}_m$ such that $\mathbf{M}_m \cap \mathbf{L}_{1,m}$ does not approach $\mathbf{M} \cap \mathbf{M}_1$ for any $\mathbf{M}$ in $\mathbf{C}$, yet $\mathbf{M}_m \cap \mathbf{L}_{1,m}$ approaches some invariant subspace $\mathbf{N}$ for the restriction of S to $\mathbf{M}_1$. The dimension of $\mathbf{M}_m \cap \mathbf{L}_{1,m}$ does not change with the choice of $\mathbf{M}_m$ in $\mathbf{C}_m$. Because all of the invariant subspaces for S in $\mathbf{M}_1$ with the same dimension as $\mathbf{N}$ are in the same connected component as $\mathbf{N}$, we have a contradiction.

Finally, the fact that isolated invariant subspaces are stable can be extended to algebraic operators on a Hilbert space of any dimension. This is a direct consequence of the characterization of the isolated invariant subspaces of an algebraic operator given in [3], the semicontinuity of separated parts of

the spectrum [7, p. 212], and the preceding theorem applied to the (necessarily finite-dimensional) cyclic generalized eigenspaces.

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DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NORTH CAROLINA 27650

DEPARTMENT OF MATHEMATICS, SWEET BRIAR COLLEGE, SWEET BRIAR, VIRGINIA 24595