AN INTEGRAL REPRESENTATION OF HOLOMORPHIC FUNCTIONS

R. W. HILGER AND J. F. MICHALIČEK

ABSTRACT. Let K be a compact set in the complex plane and let f be a function holomorphic on the complement Ω of K and vanishing at infinity. We prove that there are finite complex-valued Borel measures $\mu_{m,n}$ ($m, n = 0, 1, 2, \dots; m + n > 1$) on K^2 satisfying $\lim_{k \rightarrow \infty} (\sum_{m+n=k} \|\mu_{m,n}\|)^{1/k} = 0$ so that

$$f(z) = \sum_{m,n} \int_{K^2} (z - w_1)^{-m} (z - w_2)^{-n} d\mu_{m,n}(w_1, w_2) \quad (z \in \Omega).$$

1. Introduction. Let K be a nonempty compact set in the complex plane $\mathbb{C}$ and put $\Omega = \mathbb{C} \setminus K$. The following integral representation of holomorphic functions on Ω is due to Baernstein [1].

THEOREM 1. *If K is locally connected then every holomorphic function f on Ω vanishing at infinity has a representation*

$$f(z) = \sum_{k=1}^{\infty} \int_K (z - w)^{-k} d\mu_k(w) \quad (z \in \Omega)$$

where (μ_k) are finite complex-valued Borel measures on K satisfying $\lim_{k \rightarrow \infty} \|\mu_k\|^{1/k} = 0$.

This theorem fails if K is not locally connected. Consider e.g. the set $K = \{n^{-1} | n = 1, 2, \dots\} \cup \{0\}$ and the function

$$f(z) = \sum_{n=1}^{\infty} -n^{-2} z^{-2} (z - n^{-1})^{-1}$$

on $\Omega = \mathbb{C} \setminus K$.

In this paper we prove a generalization of Theorem 1 for arbitrary compact sets. We denote by I the set of all pairs (m, n) of nonnegative integers with $m + n > 1$ and by K^k the Cartesian product of k copies of K .

THEOREM 2. *Let f be a holomorphic function on Ω vanishing at infinity. There are finite complex-valued Borel measures $\mu_{m,n}$ on K^2 , $(m, n) \in I$, satisfying $\lim_{k \rightarrow \infty} (\sum_{m+n=k} \|\mu_{m,n}\|)^{1/k} = 0$ so that*

$$f(z) = \sum_I \int_{K^2} (z - w_1)^{-m} (z - w_2)^{-n} d\mu_{m,n}(w_1, w_2) \quad (z \in \Omega).$$

It is in general impossible to characterize a unique representation.

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COROLLARY. *Every holomorphic function on Ω vanishing at infinity has a representation*

$$f(z) = \sum_{k=1}^{\infty} \int_{K^k} ((z - w_1) \cdots (z - w_k))^{-1} d\mu_k(w_1, \dots, w_k) \quad (z \in \Omega)$$

where μ_k is a finite complex-valued Borel measure on K^k and

$$\lim_{k \rightarrow \infty} \|\mu_k\|^{1/k} = 0.$$

If K consists of one point, we obtain the Laurent expansion of f . Regarding the given example we have

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} -n^{-2} z^{-2} (z - n^{-1})^{-1} \\ &= - \int_{K^3} ((z - w_1)(z - w_2)(z - w_3))^{-1} d\mu_3(w_1, w_2, w_3) \end{aligned}$$

where μ_3 is a point measure with $\mu_3\{(0, 0, n^{-1})\} = n^{-2}$ and $z \in \Omega$.

2. Prerequisites. Given a nonempty compact set S in $\mathbb{C}$, we denote by $A(S)$ the Banach algebra of all continuous complex-valued functions on S which are holomorphic on the interior $\mathring{S}$ of S with the norm $\|f\|_S = \sup_{s \in S} |f(s)|$ ($f \in A(S)$).

Let $H(K)$ be the algebra of all germs of holomorphic functions on K . Consider a base $\{S_\lambda \mid \lambda = 1, 2, \dots\}$ of compact neighbourhoods of K whose boundaries ∂S_λ are finite unions of rectifiable Jordan curves. Suppose additionally that every component of S_λ intersects K . With the inductive topology with respect to the natural mappings $A(S_\lambda) \rightarrow H(K)$, $H(K)$ is a topological vector space. This topology is independent of the base $\{S_\lambda\}$.

We denote by $H(\Omega)$ the algebra of all holomorphic functions on Ω vanishing at infinity. Consider a sequence $(X_\lambda)_{\lambda=1,2,\dots}$ of compact sets $X_\lambda \subset \Omega$ with $X_\lambda \subset \mathring{X}_{\lambda+1}$ and $\Omega = \bigcup_{\lambda=1}^{\infty} X_\lambda$. With the projective topology with respect to the natural mappings $H(\Omega) \rightarrow A(X_\lambda)$, $H(\Omega)$ is a topological vector space. This topology is independent of (X_λ) .

Let us list some well-known properties of $H(K)$ and $H(\Omega)$ ([2], [3]). $H(K)$ and $H(\Omega)$ are Montel spaces and therefore reflexive. $H(K)$ and $H(\Omega)$ are strong duals with respect to the following bilinear form. Consider $f \in H(K)$ and $g \in H(\Omega)$. Then there exists a λ with $f \in A(S_\lambda)$.

$$\langle f, g \rangle = \frac{1}{2\pi i} \int_{\partial S_\lambda} f(z) g(z) dz$$

defines a continuous bilinear form on $H(K) \times H(\Omega)$.

Let L_ν ($\nu = 1, 2, \dots$) be the Banach space of all families $(\varphi_{m,n})_{(m,n) \in I}$ of continuous functions $\varphi_{m,n}$ on K^2 satisfying

$$\|(\varphi_{m,n})\|_\nu = \sup_{m,n} \left(\sup_{K^2} |\varphi_{m,n}| \right) \nu^{-(m+n)} < \infty$$

and

$$\lim_{m+n \rightarrow \infty} \left(\sup_{K^2} |\varphi_{m,n}| \nu^{-(m+n)} \right) = 0.$$

$L = \bigcup_{\nu=1}^{\infty} L_{\nu}$ is a topological vector space with the inductive topology with respect to the inclusions $L_{\nu} \rightarrow L$.

Let M_{ν} be the Banach space of all families $(\mu_{m,n})_{(m,n) \in I}$ of finite complex valued Borel measures on K^2 satisfying

$$\|(\mu_{m,n})\|_{\nu} = \sum_{m,n} \|\mu_{m,n}\| \nu^{m+n} < \infty.$$

$M = \bigcap_{\nu=1}^{\infty} M_{\nu}$ is a topological vector space with the projective topology with respect to the inclusions $M \rightarrow M_{\nu}$. M is the set of all families $(\mu_{m,n})_{(m,n) \in I}$ satisfying $\lim_{k \rightarrow \infty} (\sum_{m+n=k} \|\mu_{m,n}\|)^{1/k} = 0$.

Consider the bilinear form on $L_{\nu} \times M_{\nu}$ defined by

$$\langle (\varphi_{m,n}), (\mu_{m,n}) \rangle = \sum_I \int_{K^2} \varphi_{m,n}(w_1, w_2) d\mu_{m,n}(w_1, w_2).$$

Since

$$|\langle (\varphi_{m,n}), (\mu_{m,n}) \rangle| \leq \|(\varphi_{m,n})\|_{\nu} \|(\mu_{m,n})\|_{\nu}.$$

Riesz' theorem shows that M_{ν} is the strong dual space of L_{ν} . Hence the strong dual space of L is isomorphic to M [3].

$H(K)$ and L are bornological (DF)-spaces [3, p. 406], $H(\Omega)$ and M are (F)-spaces. $H(K)$ and $H(\Omega)$ are Montel spaces [2].

PROPOSITION 1 [3]. *Let E, F be locally convex spaces, E bornological; a linear mapping $T: E \rightarrow F$ is continuous if and only if $T(B) \subset F$ is bounded whenever $B \subset E$ is bounded.*

PROPOSITION 2 [1]. *Let E be a Montel space, F a (DF)-space and $T: E \rightarrow F$ a continuous linear mapping. If $T^{-1}(B)$ is bounded in E for all bounded sets $B \subset F$, then $T: E \rightarrow T(E)$ is an isomorphism of topological vector spaces.*

3. Embedding of $H(K)$ into L . Consider the linear mapping $T: H(K) \rightarrow L$ defined by

$$(Tf)_{m,n}(w_1, w_2) = \langle f, (z - w_1)^{-m}(z - w_2)^{-n} \rangle \quad (f \in H(K)).$$

THEOREM 3. *$T: H(K) \rightarrow T(H(K))$ is an isomorphism of topological vector spaces.*

PROOF. Let B be a bounded set in $H(K)$. Then B is bounded in $A(S_{\lambda})$ for some λ . For $f \in B$ we have

$$\begin{aligned} |(Tf)_{m,n}(w_1, w_2)| &= \left| \frac{1}{2\pi i} \int_{\partial S_{\lambda}} f(z)(z - w_1)^{-m}(z - w_2)^{-n} dz \right| \\ &\leq (2\pi)^{-1} |\partial S_{\lambda}| \|f\|_{S_{\lambda}} \delta^{-(m+n)}, \end{aligned}$$

where δ is the distance between K and ∂S_{λ} and $|\partial S_{\lambda}|$ is the length of ∂S_{λ} . If

we choose $\nu > \delta^{-1} + 1$ we conclude $Tf \in L_\nu$

$$\begin{aligned} \lim_{m+n \rightarrow \infty} \left(\sup_{K^2} |\varphi_{m,n}| \nu^{-(m+n)} \right) &\leq \lim_{m+n \rightarrow \infty} (2\pi)^{-1} |\partial S_\lambda| \|f\|_{S_\lambda} \delta^{-(m+n)} \nu^{-(m+n)} \\ &< \lim_{m+n \rightarrow \infty} (\nu - 1)^{(m+n)} \nu^{-(m+n)} = 0 \end{aligned}$$

and that

$$\begin{aligned} \|Tf\|_\nu &= \sup_{m,n} \left(\sup_{K^2} |(Tf)_{m,n}| \nu^{-(m+n)} \right) \\ &\leq \sup_{m,n} (2\pi)^{-1} |\partial S_\lambda| \|f\|_{S_\lambda} (\delta \cdot \nu)^{-(m+n)} \\ &\leq (2\pi)^{-1} |\partial S_\lambda| \|f\|_{S_\lambda}. \end{aligned}$$

Thus $T(B)$ is bounded in L_ν and therefore $T(B)$ is bounded in L . The continuity of T then follows from Proposition 1.

We now apply Proposition 2 to prove that $T: H(K) \rightarrow T(H(K))$ is an isomorphism. Let B be a bounded set in L . Then B is bounded in L_ν for some ν . At first we show that $T^{-1}(B) \subseteq A(S_\lambda)$ for some λ . Suppose the contrary: for every positive ε there exists a germ $f \in T^{-1}(B)$ which has no holomorphic continuation on $U_{\varepsilon/2}(K)$, the $\varepsilon/2$ -neighbourhood of K . We choose $\varepsilon < (4\nu)^{-1}$. Since $(j!)^{-1} f^{(j)}(w) = (Tf)_{1,j}(w, w)$ ($w \in K$), the power series $\sum_{j=0}^{\infty} (f^{(j)}(w)/j!)(z - w)^j$ converges for $|z - w| < \nu^{-1}$. According to our supposition there are points $w_1, w_2 \in K$, $|w_1 - w_2| < \varepsilon$, so that the power series of f in powers of $z - w_1$ and powers of $z - w_2$ are not continuations of each other. Let R_2 be a disc with center w_2 and radius $r_2 = (2\nu)^{-1}$. Let R_1 be a disc with center w_1 and radius $r_1 < |w_1 - w_2|$. The Taylor series of f in powers of $z - w_2$ defines a holomorphic function g on R_1 . If D_1 and D_2 are small discs centred at w_1, w_2 then

$$(Tf)_{m,n}(w_1, w_2) = \frac{1}{2\pi i} \int_{\partial D_1 \cup \partial D_2} f(z)(z - w_1)^{-m}(z - w_2)^{-n} dz.$$

In the second integral we expand f in powers of $z - w_2$ and use

$$\begin{aligned} \int_{\partial D_2} (z - w_1)^{-m}(z - w_2)^{-n} dz &= \int_{\partial R_2} (z - w_1)^{-m}(z - w_2)^{-n} dz \\ &\quad - \int_{\partial R_1} (z - w_1)^{-m}(z - w_2)^{-n} dz \end{aligned}$$

from which is easily deduced the equation

$$\begin{aligned} (Tf)_{m,n}(w_1, w_2) &= \frac{1}{2\pi i} \int_{\partial R_1} \sum_{j=0}^{\infty} \frac{1}{j!} (f^{(j)} - g^{(j)})(w_1)(z - w_1)^{j-m}(z - w_2)^{-n} dz \\ &\quad + \frac{1}{2\pi i} \int_{\partial R_2} \sum_{j=0}^{\infty} \frac{1}{j!} f^{(j)}(w_2)(z - w_1)^{-m}(z - w_2)^{j-n} dz. \end{aligned}$$

Let m_0 be the lowest index such that $f^{(m_0)}(w_1) \neq g^{(m_0)}(w_1)$. We fix $m = m_0 + 1$

and obtain

$$(Tf)_{m,n}(w_1, w_2) = \frac{1}{m_0!} (f^{(m_0)} - g^{(m_0)})(w_1)(w_1 - w_2)^{-n} \\ + \frac{1}{2\pi i} \int_{\partial R_2} \sum_{j=0}^{\infty} (Tf)_{1,j}(w_2, w_2)(z - w_1)^{-m}(z - w_2)^j{}^{-n} dz.$$

Note that if $z \in \partial R_2$, then $|z - w_1| > (4\nu)^{-1}$, $|z - w_2| = (2\nu)^{-1}$, and $|w_1 - w_2| < (4\nu)^{-1}$. Hence

$$|(Tf)_{m,n}(w_1, w_2)| \geq \frac{1}{m_0!} |f^{(m_0)}(w_1) - g^{(m_0)}(w_1)| 4^n \nu^n \\ - \|Tf\|_{\nu} 2^{n+2m_0+2} \nu^{n+m_0+1},$$

and we conclude $\|Tf\|_{\nu} = \infty$ which is a contradiction to $Tf \in L_{\nu}$.

Finally we show that $T^{-1}(B)$ is bounded in $A(S_{\lambda})$. Suppose $S_{\lambda} \subseteq U_{1/2\nu}(K)$. For every $z \in S_{\lambda}$ there exists a $w \in K$ so that $|z - w| < (2\nu)^{-1}$. Then for all $f \in T^{-1}(B)$ we have

$$f(z) = \sum_{j=0}^{\infty} (Tf)_{1,j}(w, w)(z - w)^j.$$

Hence

$$\|f\|_{S_{\lambda}} \leq \sup_{z \in S_{\lambda}} \sum_{j=0}^{\infty} |(Tf)_{1,j}(w, w)| |z - w|^j \leq \nu \|Tf\|_{\nu}.$$

Thus $T^{-1}(B)$ is bounded in $A(S_{\lambda})$ and therefore $T^{-1}(B)$ is bounded in $H(K)$. Proposition 2 completes the proof.

COROLLARY. *The adjoint mapping $T^*: L^* = M \rightarrow H(K)^* = H(\Omega)$ is surjective.*

PROOF OF THEOREM 2. Suppose $f \in H(\Omega)$. The last corollary shows the existence of $(\mu_{m,n}) \in M$ such that $T^*(\mu_{m,n}) = f$. Let

$$\varphi(z) = \sum_I \int_{K^2} (z - w_1)^{-m}(z - w_2)^{-n} d\mu_{m,n}(w_1, w_2).$$

Then $\varphi \in H(\Omega)$ and, by use of Fubini's Theorem, we deduce $\langle h, f \rangle = \langle Th, (\mu_{m,n}) \rangle = \langle h, \varphi \rangle$, for all $h \in H(K)$. It follows that $f = \varphi$.

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UNIVERSITY OF HAMBURG, 2 HAMBURG 13, FEDERAL REPUBLIC OF GERMANY