CORRECTION OF A GENERALIZATION OF A THEOREM OF BEURLING AND LIVINGSTON

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ABSTRACT. A generalization of a Riesz-Fischer theorem proved by Beurling and Livingston for smooth uniformly convex Banach spaces also holds for smooth, strictly convex, reflexive Banach spaces. **THEOREM.** *Let B be a smooth, strictly convex, reflexive Banach space. Let $T: B \rightarrow B^*$ be a duality map, C a closed subspace of B , $h \in B$, $k \in B^*$. Then $T(C + h) \cap (C^\perp + k)$ is a single point. A two-dimensional counterexample shows that $T(C + h) \cap (C^\perp + k) = \emptyset$ is possible if B is not smooth, contrary to the claim of Theorem 4 of Browder, *On a theorem of Beurling and Livingston*, Canad. J. Math. 17 (1965), 367–372.*

Introduction. Beurling and Livingston [1, Theorem 2], presented a generalization of the Riesz-Fischer theorem using single-valued duality map from a uniformly convex, smooth Banach space B to its conjugate space B^* . The paper [2, Theorem 4] attempted to extend the result to single-valued duality maps on strictly convex reflexive Banach spaces without the hypothesis of smoothness. This extended result was cited in the paper [3, p. 338], which dealt primarily with multi-valued duality maps. The present paper demonstrates with a counterexample that the claim of [2] is in error, but that if the hypothesis of smoothness is restored, then the theorem can be proved along the lines of [2].

1. Preliminaries. We follow the definition and notation of [2, §2]. Denote by B a Banach space and let $S = \{u \in B \mid \|u\| = 1\}$ be the unit sphere.

DEFINITION. A *duality map* is a single-valued map $T: B \rightarrow B^*$ which verifies

$$\langle u, T(u) \rangle = \|u\| \cdot \|T(u)\| \quad \text{for all } u \in B,$$

and for some continuous, strictly increasing $\phi: R^+ \rightarrow R^+$ with $\phi(0) = 0$ and $\lim_{r \rightarrow \infty} \phi(r) = \infty$,

$$\|T(u)\| = \phi(\|u\|) \quad \text{for all } u \in B.$$

Recall that a Banach space is *smooth* at $u \neq 0$ if S has a unique supporting hyperplane at $u/\|u\|$, or equivalently, the Gateaux differential

$$G(w, u) = \lim_{t \rightarrow 0} \frac{\|u + tw\| - \|u\|}{t}$$

exists for all $w \in B$. See for example James [4, p. 239]. It is immediate that for a given ϕ , a Banach space is smooth at $u \neq 0$ iff the value at u of any

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duality map corresponding to ϕ is uniquely determined by the formula

$$\langle w, T(u) \rangle = \phi(\|u\|) \cdot G(w, u) \quad \text{for all } w \in B.$$

Thus, a Banach space is smooth (at all $u \neq 0$) iff the duality map corresponding to a given ϕ is unique.

2. Counterexample. It follows easily from the above discussion that part (b) of Lemma 2.3 in [2] is false. Also, part (c) of Lemma 2.3 and Theorem 4 in [2] are false, as is shown below.

Let $B = R^2$, $h = (1, 0) \in R^2$. Let $\|\cdot\|$ denote any norm for which $\pm h \in S$ and S is not smooth at $\pm h$. Further, require that $\|\cdot\|$ be chosen so $\pm h$ are the only points of S at which supporting lines for S are parallel to the y -axis. (For example, the strictly convex norm

$$\|(x, y)\| = |y| + \sqrt{x^2 + y^2}$$

gives a unit sphere $S = \{(x, y) | y = \pm (\frac{1}{2} - \frac{1}{2}x^2), -1 \leq x \leq 1\}$ with these properties.) For $\phi(r) = r$, let T be any corresponding duality map for which $L = \{u \in R^2 | \langle u, T(h) \rangle = 1\}$ is a supporting line for S at h other than the line $x = 1$. Let C be the y -axis and k the zero of B^* . Then $C + h$ is the line $x = 1$.

We assert that

$$T(C + h) \cap (C^\perp + k) = T(C + h) \cap C^\perp = \emptyset,$$

which shows Theorem 4 in [2] to be false. In fact, $T(h) \notin C^\perp$ because $T(h) \in C^\perp$ implies $\langle c + h, T(h) \rangle = \langle h, T(h) \rangle = 1$ for any $c \in C$. This is impossible since T was chosen so that $C + h \neq L$. If $T(c + h) \in C^\perp$ for any nonzero $c \in C$, then

$$\begin{aligned} \|c + h\|^2 &= \langle c + h, T(c + h) \rangle = \langle h, T(c + h) \rangle \\ &\leq \|h\| \cdot \|T(c + h)\| = \|c + h\| \end{aligned}$$

which entails $\|c + h\| \leq 1$. But $\|c + h\| > 1$ must hold because $\pm h$ are the only points of S at which supporting lines are parallel to C . Thus $T(c + h) \notin C^\perp$, and the assertion is proved.

Further, it is clear that none of the infinitely many duality maps corresponding to $\phi(r) = r$ is continuous as a map into B^* with the weak topology. Thus, Lemma 2.3(c) of [2] is false.

3. Corrections. The origin of the error in [2, Theorem 4] lies in the errors in [2, Lemma 2.3(b), (c)]. Corrected statements of (b) and (c) and of this theorem follow.

(b) A Banach space B is smooth iff there is exactly one duality map T corresponding to each function ϕ of the Definition.

(c) If B is reflexive and smooth then the duality map T corresponding to each ϕ is a continuous mapping of B into B^* with the weak topology.

THEOREM (BEURLING-LIVINGSTON). *Let B be a reflexive, strictly convex and smooth Banach space, T a duality map of B into B^* . Let C be a closed subspace of B , h a fixed element of B , k a fixed element of B^* . Then the set*

$$T(C + h) \cap (C^\perp + k)$$

contains exactly one element.

That (b) is true follows from the discussion in §1. The proof of (c) follows from (b) in a manner analogous to the argument in [2, p. 371]. The proof of the Theorem is then analogous to the argument in [2, p. 372].

REFERENCES

1. A. Beurling and A. E. Livingston, *A theorem on duality mappings in Banach spaces*, Ark. Mat. **4** (1961), 405–411.
2. F. E. Browder, *On a theorem of Beurling and Livingston*, Canad. J. Math. **17** (1965), 367–372.
3. ———, *Multivalued monotone nonlinear mappings and duality mappings in Banach space*, Trans. Amer. Math. Soc. **118** (1965), 338–351.
4. R. C. James, *Linear functionals as differentials of a norm*, Math. Mag. **24** (1951), 237–244.

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