ON $\text{Ext}_{K_*(K)}(\pi_*(K), K_*(MU))$

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ABSTRACT. In this note a spectral sequence is constructed which is used to prove the existence of elements of nonzero homological degree in $\text{Ext}_{K_*(K)}(\pi_*(K), K_*(MU))$. Examples of specific elements in terms of representatives in the cobar construction are given.

1. Introduction and statement of results. If E is a multiplicative spectrum such that $E_*(E)$ is a flat $\pi_*(E)$ -module, then, under certain conditions listed in [2, pp. 317–318], there is, for spectra X and Y , a spectral sequence

$$\text{Ext}_{E_*^*(E)}^{p,*}(E_*(X), E_*(Y)) \Rightarrow [X, Y]_*^E,$$

where $[X, Y]_*^E$ denotes the set of morphisms in a certain category of spectra. Roughly speaking, $[X, Y]^E$ is $[X, Y]$ modulo those maps which E -theory cannot detect. The Ext groups are computed in the category of comodules over $E_*(E)$.

Taking $E = K$, $X = S^0$, $Y = MU$ we have a spectral sequence whose E_2 -term is $\text{Ext}_{K_*(K)}^{p,*}(\pi_*(K), K_*(MU))$. However, K is not connective, and so assumption (iii)(b) on p. 317 of [2] does not hold; thus we do not have the convergence part of Theorem 15.1 of Part III of [2]. On the other hand the edge-homomorphism

$$\pi_*(MU) \rightarrow \text{Ext}_{K_*(K)}^{0,*}(\pi_*(K), K_*(MU)) = PK_*(MU)$$

is an isomorphism; this is equivalent to the Hattori-Stong theorem; see [2, Theorem 14.1 of Part II] or [7, Theorem 20.34].

It seemed natural to suspect that

$$\text{Ext}_{K_*(K)}^{p,*}(\pi_*(K), K_*(MU)) = 0 \quad \text{for } p \neq 0$$

and that the spectral sequence collapsed. The hope was that, if one could prove this, the convergence problem might be circumvented and new proofs of the Hattori-Stong theorem and Milnor's theorem on the structure of $\pi_*(MU)$ obtained. See [7, pp. 516–517].

The purpose of this note is to prove that this conjecture is false. That is, we show that $\text{Ext}_{K_*(K)}(\pi_*(K), K_*(MU))$ contains nonzero elements of positive homological degree. Our proof involves considering $K_*(MU)$ at the same time as a left $K_*(K)$ -comodule and as a right $MU_*(MU)$ -comodule.

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THEOREM 1. *There is a (trigraded) spectral sequence with E_2 -term*

$$\text{Ext}_{MU_*(MU)}^{p,*}(\pi_*(MU), \text{Ext}_{K_*(K)}^{q,*}(\pi_*(K), K_*(MU)))$$

which converges to $\text{Ext}_{K_(K)}^{p+q,*}(\pi_*(K), \pi_*(K))$.*

Here the E_2 -term is calculated as follows. The inner Ext groups are computed in the category of left $K_*(K)$ -comodules; each $\text{Ext}_{K_*(K)}^{q,*}(\pi_*(K), K_*(MU))$ is a right $MU_*(MU)$ -comodule and the outer Ext groups are defined in the category of such comodules. More details of this algebra will be found below.

COROLLARY. $\text{Ext}_{K_*(K)}^{q,*}(\pi_*(K), K_*(MU))$ is nonzero for some positive q .

PROOF OF COROLLARY FROM THEOREM 1. If the Corollary is false the spectral sequence of the Theorem has

$$E_2^{p,q} = \begin{cases} \text{Ext}_{MU_*(MU)}^{p,*}(\pi_*(MU), \pi_*(MU)), & q = 0, \\ 0, & q \neq 0. \end{cases}$$

(Note that this Ext group is taken in the category of right $MU_*(MU)$ -comodules; but it is clearly isomorphic to the usual E_2 -term of the Adams-Novikov spectral sequence.)

Now the spectral sequence collapses giving

$$\text{Ext}_{MU_*(MU)}^{p,*}(\pi_*(MU), \pi_*(MU)) \cong \text{Ext}_{K_*(K)}^{p,*}(\pi_*(K), \pi_*(K)).$$

But this contradicts various calculations made of these groups. For example,

$$\text{Ext}_{MU_*(MU)}^{1,k}(\pi_*(MU), \pi_*(MU)) = 0 \quad \text{for } k < 0,$$

while

$$\text{Ext}_{K_*(K)}^{1,2n}(\pi_*(K), \pi_*(K)) = \mathbf{Z}/m(n)\mathbf{Z} \quad \text{for all } n \in \mathbf{Z} \quad [5].$$

Alternatively,

$$\text{Ext}_{MU_*(MU)}^{1,4}(\pi_*(MU), \pi_*(MU)) = \mathbf{Z}/12\mathbf{Z} \quad [6], [8].$$

while

$$\text{Ext}_{K_*(K)}^{1,4}(\pi_*(K), \pi_*(K)) = \mathbf{Z}/24\mathbf{Z}.$$

We could also appeal to the theorem of Adams and Baird [3], and [8].

These discrepancies lead one to look for specific nonzero elements in $\text{Ext}_{K_*(K)}^{1,*}(\pi_*(K), K_*(MU))$ in terms of their representatives in the cobar construction. We give examples of such elements in §3 and will return to this and other related questions in a later paper.

2. Proof of Theorem 1. $K_*(MU)$ will be considered as a left $K_*(K)$ -comodule and as a right $MU_*(MU)$ -comodule with structure maps

$$K_*(MU) \xrightarrow{\psi} K_*(K) \otimes_{\pi_*(K)} K_*(MU)$$

and

$$K_*(MU) \xrightarrow{\psi'} K_*(MU) \otimes_{\pi_*(MU)} MU_*(MU).$$

Here $\psi = \psi_{MU}$ is the usual map of [1, Lecture 3], and ψ' is defined by the same construction, but with everything switched from left to right throughout. Thus ψ' is the composition

$$\begin{aligned} K_*(MU) &\xrightarrow{t_*} MU_*(K) \xrightarrow{\psi_K} MU_*(MU) \otimes_{\pi_*(MU)} MU_*(K) \\ &\cong K_*(MU) \otimes_{\pi_*(MU)} MU_*(MU), \end{aligned}$$

where the last isomorphism sends $\alpha \otimes \beta$ to $t^*\beta \otimes c\alpha$, c being the canonical antiautomorphism of $MU_*(MU)$, and t the switch map.

The following diagram commutes:

$$\begin{array}{ccc} K_*(MU) & \xrightarrow{\psi} & K_*(K) \otimes_{\pi_*(K)} K_*(MU) \\ \psi' \downarrow & & \downarrow 1 \otimes \psi' \\ K_*(MU) \otimes_{\pi_*(MU)} MU_*(MU) & \xrightarrow{\psi \otimes 1} & K_*(K) \otimes_{\pi_*(K)} K_*(MU) \otimes_{\pi_*(MU)} MU_*(MU) \end{array}$$

That is, ψ is a morphism of right $MU_*(MU)$ -comodules and ψ' is a morphism of left $K_*(K)$ -comodules.

LEMMA.

$$\text{Ext}_{MU_*(MU)}^{p,*}(\pi_*(MU), MU_*(K)) = \begin{cases} \pi_*(K), & p = 0, \\ 0, & p \neq 0. \end{cases}$$

PROOF. By the Conner-Floyd theorem, $MU_*(K) = MU_*(MU) \otimes_{\pi_*(MU)} \pi_*(K)$; thus $MU_*(K)$ is an extended $MU_*(MU)$ -comodule and the lemma follows [2, p. 321] or [7, Chapter 19].

It follows that, in terms of right $MU_*(MU)$ -comodules,

$$\text{Ext}_{MU_*(MU)}^{p,*}(\pi_*(MU), K_*(MU)) = \begin{cases} \pi_*(K), & p = 0, \\ 0, & p \neq 0. \end{cases}$$

If M is a left comodule over a two-sided coalgebra C with ground ring R , then $\text{Ext}_C(R, M)$ may be computed as the homology of the cobar complex

$$M \xrightarrow{d_0} \bar{C} \otimes_R M \xrightarrow{d_1} \bar{C} \otimes_R \bar{C} \otimes_R M \xrightarrow{d_2} \cdots,$$

where $\bar{C} = \ker\{\varepsilon: C \rightarrow R\}$ and

$$\begin{aligned} d_i(c_1 \otimes \cdots \otimes c_i \otimes m) &= 1 \otimes c_1 \otimes \cdots \otimes c_i \otimes m + (-1)^{i+1} c_1 \\ &\otimes \cdots \otimes c_i \otimes \psi m + \sum_{j=1}^i (-1)^j c_1 \otimes \cdots \otimes \psi c_j \otimes \cdots \otimes c_i \otimes m. \end{aligned}$$

Similarly, we can form a cobar resolution for right comodules. Let us write $M = K_*(MU)$, $C = K_*(K)$, $R = \pi_*(K)$, $D = MU_*(MU)$ and $S = \pi_*(MU)$.

Since the structure maps $M \xrightarrow{\psi} C \otimes_R M$ and $M \xrightarrow{\psi} M \otimes_S D$ commute, we may put both cobar complexes together to form a double complex.

$$\begin{array}{ccccccc}
 \vdots & & \vdots & & \vdots & & \\
 \bar{C} \otimes_R \bar{C} \otimes_R M & \rightarrow & \bar{C} \otimes_R \bar{C} \otimes_R M \otimes_S \bar{D} & \rightarrow & & \rightarrow & \dots \\
 \uparrow & & \uparrow & & \uparrow & & \\
 \bar{C} \otimes_R M & \rightarrow & \bar{C} \otimes_R M \otimes_S \bar{D} & \rightarrow & \bar{C} \otimes_R M \otimes_S \bar{D} \otimes_S \bar{D} & \rightarrow & \dots \\
 \uparrow & & \uparrow & & \uparrow & & \\
 M & \rightarrow & M \otimes_S \bar{D} & \rightarrow & M \otimes_S \bar{D} \otimes_S \bar{D} & \rightarrow & \dots
 \end{array}$$

Now a double complex A has associated with it two spectral sequences, whose E -terms are $H_I H_{II} A$ and $H_{II} H_I A$, respectively (where H_I represents homology with respect to the horizontal differentials and H_{II} with respect to the vertical differentials), and both of which converge to the total homology $H(A)$ [4, pp. 331–332]. We will see that the second spectral sequence collapses, allowing us to compute $H(A)$, and then the first spectral sequence will be that of Theorem 1.

C and D , and hence $\bar{C}$ and $\bar{D}$, are flat modules over R and S . By the Lemma, the first row is exact, and so all the rows are exact and, thus,

$$H_I(A)^{p,q} = \begin{cases} \bar{C} \otimes_R \cdots \otimes_R \bar{C} & (q \text{ factors}), & p = 0, \\ 0, & p \neq 0, \end{cases}$$

giving

$$H_{II} H_I(A)^{p,q} = \begin{cases} \text{Ext}_C^q(R, R), & p = 0, \\ 0, & p \neq 0. \end{cases}$$

We must therefore have $H(A) = \text{Ext}_C^*(R, R)$.

Turning to the first spectral sequence, we know that the first column has $\text{Ext}_C^*(R, M)$ as its homology. Hence by the flatness of $\bar{D}$, the q th column has homology $\text{Ext}_C^*(R, M) \otimes_S \bar{D} \otimes_S \cdots \otimes_S \bar{D}$ (q factors). It follows that the E_2 -term of the first spectral sequence is $\text{Ext}_D^p(S, \text{Ext}_C^q(R, M))$. Restoring the notation gives Theorem 1.

3. Nonzero elements in $\text{Ext}_{K_*(K)}^1(\pi_*(K), K_*(MU))$. We show here that, in the standard notation of [2] and [7],

$$(u^n - v^n)/m(n) \otimes 1 \in \overline{K_*(K)} \otimes_{\pi_*(K)} K_*(MU)$$

is a cycle in the cobar complex for computing $\text{Ext}_{K_*(K)}(\pi_*(K), K_*(MU))$, which, for n negative, gives an element of order $m(n)$ in $\text{Ext}^{1,2n}$, and, for $n = 2$, gives an element of order 2 in $\text{Ext}^{1,4}$.

We can see this as follows. $(u^n - v^n)/m(n) \otimes 1$ is a cycle because $(u^n - v^n)/m(n) \in K_{2n}(K)$ is primitive [5].

$$d_0: K_*(MU) \rightarrow \overline{K_*(K)} \otimes_{\pi_*(K)} K_*(MU)$$

is defined by $d_0\alpha = \psi\alpha - 1 \otimes \alpha$. Write

$$K_*(MU) = \pi_*(K)[b_1, b_2, \dots] = \mathbb{Z}[t, t^{-1}][b_1, b_2, \dots];$$

then $d_0t^n = (u^n - v^n) \otimes 1$.

Now suppose that $k[(u^n - v^n)/m(n) \otimes 1] = 0$ in $\text{Ext}^{1,2n}$, so that there exists $\alpha \in K_*(MU)$ such that

$$d_0\alpha = k(u^n - v^n)/m(n) \otimes 1.$$

Then $d_0(m(n)\alpha - kt^n) = 0$, and $m(n)\alpha - kt^n$ is an element of $PK_{2n}(MU)$, nonzero unless $\alpha = kt^n/m(n)$, and k is a multiple of $m(n)$. But if n is negative, $PK_{2n}(MU) \cong \pi_{2n}(MU) = 0$.

Similarly, $[(u^2 - v^2)/24 \otimes 1]$ has order 2 in $\text{Ext}^{1,4}$. For Proposition 17.38 of [7] gives

$$d_0(2b_2 - b_1^2 + tb_1) = (v^2 - u^2)/12 \otimes 1.$$

If there exists $\alpha \in K_4(MU)$ such that $d_0\alpha = (u^2 - v^2)/24 \otimes 1$, then, as before, $2\alpha + 2b_2 - b_1^2 + tb_1 \in PK_4(MU)$. But (see [7, p. 437]) $PK_4(MU)$ is generated by $3b_2 - 2b_1^2 + tb_1$ and $(2b_1 + t)^2$. Then we have, for some integers A and B ,

$$2\alpha + 2b_2 - b_1^2 + tb_1 = A(3b_2 - 2b_1^2 + tb_1) + B(2b_1 + t)^2,$$

and the coefficient of b_1^2 in 2α is $4B - 2A + 1$, giving a contradiction. Hence $[(u^2 - v^2)/24 \otimes 1]$ has order 2.

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