ON PROJECTIONS OF FINITELY ADDITIVE MEASURES

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ABSTRACT. A theorem of Z. Frolík and M. E. Rudin states that for every two-valued measure μ on N , if $F: N \rightarrow N$ is such that $F_*(\mu) = \mu$ then $F(x) = x$ for almost all x . We prove that a generalization of this theorem fails for measures in general: **THEOREM.** *There exist a translation invariant measure μ on N and a function $F: N \rightarrow N$ such that $F_*(\mu) = \mu$, and if $A \subseteq N$ is such that F is one-to-one on A , then $\mu(A) < \frac{1}{2}$.*

Let $N = \{0, 1, 2, \dots\}$ denote the set of all natural numbers. A (finitely additive) *measure* on N is a function $\mu: \mathcal{P}(N) \rightarrow [0, 1]$ which satisfies

- (a) $\mu(\emptyset) = 0$ and $\mu(N) = 1$,
- (b) if X and Y are disjoint subsets of N

$$\text{then } \mu(X \cup Y) = \mu(X) + \mu(Y).$$
(1)

A measure μ is *two-valued* if 0 and 1 are the only values of μ ; μ is *nonprincipal* if $\mu(E) = 0$ for every finite set $E \subset N$.

Let $F: N \rightarrow N$ be a function. If μ is a measure on N , then $\nu = F_*(\mu)$ (the *projection* of μ by F) is the measure defined by

$$\nu(X) = \mu(F^{-1}(X)). \quad (2)$$

The purpose of this paper is to prove the following

THEOREM. *There exists a translation invariant measure μ on N and a function $F: N \rightarrow N$ such that*

- (a) $F_*(\mu) = \mu$;
- (b) *if $A \subseteq N$ is such that F is one-to-one on A , then $\mu(A) < \frac{1}{2}$.*

This theorem contrasts with the following theorem of Z. Frolík and M. E. Rudin (see [1] and [2]):

THEOREM. *If μ is a two-valued measure on N and if $F: N \rightarrow N$ is such that $F_*(\mu) = \mu$, then $F(x) = x$ on a set of measure 1.*

To prove our theorem, we shall first find the measure μ .

If X is a set of natural numbers, and $k \in N$, we let

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$$X + k = \{n + k : n \in X\}, \quad kX = \{k \cdot n : n \in X\}. \quad (3)$$

Let $\{a_n : n \in N\}$ be a bounded sequence of real numbers, and let ν be a two-valued measure on N . Then there exists a unique real number a , which we denote

$$a = \lim_{\nu} a_n$$

such that for every $\varepsilon > 0$ there exists X with $\nu(X) = 1$ such that $|a - a_n| < \varepsilon$ whenever $n \in X$. If ν is nonprincipal then a is an accumulation point of the sequence $\{a_n : n \in N\}$.

LEMMA. *There exists a measure μ on N such that for every $X \subseteq N$,*

$$(a) \mu(X + 1) = \mu(X), \text{ and}$$

$$(b) \mu(2X) = \frac{1}{2} \mu(X).$$

PROOF. For each $X \subseteq N$ and each $n \in N$, let

$$X(n) = \text{the number of elements of } X \cap \{0, \dots, n-1\}. \quad (4)$$

As a first step, we pick an arbitrary nonprincipal two-valued measure ν and define, for each $X \subseteq N$,

$$\mu_0(X) = \lim_{\nu} X(n)/n. \quad (5)$$

It is easy to verify that μ_0 is a nonprincipal measure on N , that μ_0 satisfies

$$\mu_0(X + 1) = \mu_0(X) \quad (6)$$

for all X , and that for every $k \in N$,

$$\mu_0(kN) = 1/k. \quad (7)$$

Next we consider a sequence of measures μ_n , $n \geq 1$, defined as follows:

$$\mu_n(X) = \frac{1}{n} \cdot \sum_{k=0}^{n-1} 2^k \cdot \mu_0(2^k X). \quad (8)$$

It is clear that each μ_n is a nonprincipal measure on N : the only property not immediately clear is $\mu_n(N) = 1$, but that follows from (7). Also, since μ_0 satisfies (6), each μ_n satisfies

$$\mu_n(X + 1) = \mu_n(X) \quad (9)$$

for all $X \subseteq N$.

Now we define, for all $X \subseteq N$,

$$\mu(X) = \lim_{\nu} \mu_n(X). \quad (10)$$

Again, μ is a nonprincipal measure on N , and

$$\mu(X + 1) = \mu(X) \quad (11)$$

for all $X \subseteq N$. We will complete the proof of Lemma by showing that for all X ,

$$\mu(2X) = \frac{1}{2} \mu(X). \quad (12)$$

Let $X \subseteq N$ be arbitrary. For each $n \geq 1$ we have

$$\begin{aligned} |\mu_0(X) - 2\mu_n(2X)| &= \left| \frac{1}{n} \cdot \sum_{k=0}^{n-1} 2^k \mu_0(2^k X) - \frac{2}{n} \sum_{k=0}^{n-1} 2^k \mu_0(2^{k+1} X) \right| \\ &= \frac{1}{n} \cdot |\mu_0(X) - 2^n \mu_0(2^n X)| \leq \frac{1}{n} \end{aligned} \quad (13)$$

because $\mu_0(X) \leq 1$ and $\mu_0(2^n X) \leq \mu_0(2^n N) = 1/2^n$ by (7).

Applying $\lim_n$ to both sides of (13), we get $|\mu(X) - 2\mu(2X)| = 0$ and (12) follows. $\square$

PROOF OF THEOREM. Let us define the function F . Each natural number $m > 0$ can be written uniquely as

$$m = 2^n \cdot (2k + 1). \quad (14)$$

Let $F(m)$ be the unique k such that (14) holds (and $F(0) = 0$). The function F satisfies

$$F(2k + 1) = k \quad (15)$$

and, for all $m \in N$,

$$F(2m) = F(m). \quad (16)$$

We will show that F has the properties (a) and (b) in the Theorem.

For each $n \in N$, let

$$A_n = 2^n(2N + 1). \quad (17)$$

The sets A_n are pairwise disjoint, and for each n , $\mu(A_n) = 1/2^{n+1}$. Let $B_n = \bigcup_{k=0}^n A_k$; hence $\mu(B_n) = 1 - 1/2^{n+1}$.

To prove the property (a) of the Theorem, let $X \subseteq N$; we will show that $\mu(f_{-1}(X)) = \mu(X)$.

It follows from the definition of F that for every $n \in N$,

$$f_{-1}(X) \cap A_n = 2^n(2X + 1). \quad (18)$$

Consequently, if we denote $a = \mu(X)$, and if $n \in N$, then

$$\mu(f_{-1}(X) \cap B_n) = a \cdot (1 - 1/2^{n+1}) \quad (19)$$

and

$$\mu(B_n - f_{-1}(X)) = (1 - a)(1 - 1/2^{n+1}). \quad (20)$$

Now if n tends to infinity, (19) and (20) lead to $\mu(f_{-1}(X)) = a$, which proves the claim.

To prove the property (b) of the Theorem, let $X \subseteq N$ be such that $F(x) \neq F(y)$ whenever $x, y \in X$ and $x \neq y$; we will show that $\mu(X) \leq \frac{1}{2}$.

Let $a = \mu(X)$ and suppose, without loss of generality, that $0 \notin X$. From (16) and because F is one-to-one on X it follows that the sets

$$X, 2X, 4X, \dots, 2^n X, \dots \quad (21)$$

are mutually disjoint. Moreover, $\mu(X) = a$, $\mu(2X) = a/2, \dots, \mu(2^n X) = a/2^n$, etc. Hence if $a > \frac{1}{2}$, then there is n such that $a + a/2 + \dots + a/2^n > 1$ and since the sets (21) are disjoint, we have

$$\mu(X \cup \dots \cup 2^n X) = a + a/2 + \dots + a/2^n > 1,$$

a contradiction.

The proof of the Theorem is thus complete.

REMARKS. 1. The measure μ constructed above has the property that for each $X \subseteq N$,

$$\liminf \frac{X(n)}{n} \leq \mu(X) \leq \limsup \frac{X(n)}{n}.$$

(One proves first, by induction on k , that each μ_k has this property, using the fact that

$$\limsup \frac{(2X)(n)}{n} \leq \frac{1}{2} \limsup \frac{X(n)}{n},$$

and similarly for $\liminf$.)

A set $X \subseteq N$ is said to have a density if the limit $d(X) = \lim X(n)/n$ exists. Thus if $X \subseteq N$ has a density, then $\mu(X) = d(X)$.

2. By a modification of the proof of Lemma, one can construct a measure on N such that for every $X \subseteq N$,

(a) $\mu(X + 1) = \mu(X)$, and

(b) $\mu(kX) = \mu(X)/k$, for all $k \geq 1$.

This enables one to improve " $\mu(A) \leq \frac{1}{2}$ ", in part (b) of the Theorem, to " $\mu(A) < \varepsilon$ ". K. Kunen and M. Yasumoto have improved this to " $\mu(A) = 0$ ".

The following remarks were made by the referee. In set theory without the axiom of choice, assuming the existence of a nonprincipal ultrafilter is known to be strictly stronger than assuming the existence of a nonprincipal measure; see [3]. The results in this paper require only the assumption that there is a nonprincipal measure on N . Note that the "ultrafilter limit" of a bounded sequence of real numbers is really the integral with respect to the measure ν , and all the constructions in this paper work just as well if ν is any nonprincipal real-valued measure. The measure μ defined by formulas (8) and (10) could be described more simply as

$$\mu(X) = \int 2^m \cdot \mu_0(2^m X) d\mu_0(m).$$

Since $\mu_0(X) = \mu_0(X + 1)$ for every $X \subseteq N$, it follows that

$$\int f(m + 1) d\mu_0(m) = \int f(m) d\mu_0(m)$$

for every bounded function $f: N \rightarrow R$; hence

$$\begin{aligned}
 \mu(2X) &= \int 2^m \cdot \mu_0(2^{m+1}X) \, d\mu_0(m) \\
 &= \frac{1}{2} \int 2^{m+1} \cdot \mu_0(2^{m+1}X) \, d\mu_0(m) \\
 &= \frac{1}{2} \int 2^m \cdot \mu_0(2^mX) \, d\mu_0(m) = \frac{1}{2} \mu(X).
 \end{aligned}$$

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