PSEUDO-OPEN MAPPINGS FROM TOPOLOGICAL SUMS

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ABSTRACT. Assume S is the free topological sum of spaces X_1 and X_2 , on the same set X , and $\phi: S \rightarrow X$ is the canonical quotient map. The main result states that X is developable (resp. quasi-developable) if X_1 and X_2 are regular developable (resp. quasi-developable) spaces and ϕ is pseudo-open. Examples are given to show that certain topological properties are not preserved under such a construction.

1. Introduction. Suppose Λ is a finite family of topologies on the same set X . The family Λ is said to be *compatible* [A] if for every $A \subseteq X$,

$$\overline{A}_{\cap \Lambda} = \bigcup \{ \overline{A}_\tau : \tau \in \Lambda \}.$$

If S is the free topological sum of the spaces (X, τ) , $\tau \in \Lambda$, and $\phi: S \rightarrow X$ is the canonical mapping, the compatibility condition on Λ can be stated in terms of pseudo-openness of the mapping ϕ as follows.

1.1 PROPOSITION [A]. *A finite family Λ of topologies on X is compatible if and only if the canonical map ϕ from the free topological sum S onto $(X, \cap \Lambda)$ is pseudo-open.*

In any case, it is easy to see that the intersection topology $\cap \Lambda$ on X agrees with the quotient topology on X induced by the mapping $\phi: S \rightarrow X$. This gives a slightly different version of Proposition 1.1.

1.2 PROPOSITION. *If Λ is a finite family of topologies on X then Λ is a compatible family if and only if the canonical quotient mapping $\phi: S \rightarrow X$ is pseudo-open.*

If each (X, τ) , $\tau \in \Lambda$, has a certain topological property it is natural to ask whether $(X, \cap \Lambda)$ has the same topological property (especially when Λ is compatible). It is clear from Proposition 1.2 that such questions can be studied within the framework of mappings. In this note we answer three questions of Arhangel'skiĭ [A] about the intersection of compatible topologies, with the main result saying that the intersection of two regular developable compatible topologies on X is a developable topology. We show this is not necessarily true if one of the topologies is only Hausdorff and give other

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examples showing that certain topological properties are not preserved under the intersection of two compatible topologies.

In the remainder of this paper X_1 and X_2 represent topological spaces on the same set X and S is the free topological sum of X_1 and X_2 . The mapping $\phi: S \rightarrow X$ is the natural canonical mapping and when X is used in this fashion (as the range of ϕ) we assume X has the quotient topology induced by ϕ . We will not usually distinguish between the sets X_1 , X_2 and their disjoint copies used in constructing S . This should cause no confusion, but if necessary it may be helpful to use the notation: $X_1 = X \times \{1\}$, $X_2 = X \times \{2\}$, and $S = X_1 \cup X_2$. All topological spaces considered are at least T_1 , and N denotes the set of natural numbers.

2. Intersection of developable compatible topologies. Recall that a space X is *quasi-developable* if there is a sequence $\{\mathcal{G}_n\}_1^\infty$ of families of open subsets of X such that whenever $x \in U \subset X$, where U is open in X , there is some $n \in N$ such that $x \in \text{St}(x, \mathcal{G}_n) \subset U$. X is developable if and only if X is quasi-developable and perfect (i.e., all closed subsets are G_δ sets) [Be]. Bennett and Lutzer have shown [BL, proof of Theorem 8] that X is quasi-developable if and only if there is a quasi-development $\{\mathcal{G}_n\}_1^\infty$ for X such that whenever $x \in U$ and U is open, there is some n such that $x \in \text{St}(x, \mathcal{G}_n) \subset U$ and $\text{ord}(x, \mathcal{G}_n) = 1$ (where $\text{ord}(x, \mathcal{G}_n)$ denotes the number of elements of $\mathcal{G}_n$ which contain x). For each $G \in \mathcal{G}_n$, let $F(G) = \{x \in G: \text{ord}(x, \mathcal{G}_n) = 1\}$ and let $\mathcal{P}_n$ be the collection of all pairs $(F(G), G)$, for $G \in \mathcal{G}_n$. If $\mathcal{P} = \bigcup_{n=1}^\infty \mathcal{P}_n$ we have provided part of the construction necessary for the proof of the next result. The rest of the proof will be left to the reader.

2.1 PROPOSITION. *A space X is quasi-developable if and only if there is a family $\mathcal{P} = \bigcup_{n=1}^\infty \mathcal{P}_n$ of pairs of subsets of X satisfying:*

- (a) *If $n \in N$ and $P = (P', P'') \in \mathcal{P}_n$ then $P' \subset P''$, P'' is open in X , and $P'' \cap P'_1 = \emptyset$ for any $(P'_1, P''_1) \in \mathcal{P}_n$, $(P'_1, P''_1) \neq (P', P'')$.*
- (b) *If $x \in U \subset X$, where U is open in X , there is some $P = (P', P'') \in \mathcal{P}$ such that $x \in P' \subset P'' \subset U$.*

A collection $\mathcal{P}$, of pairs of subsets satisfying (b) of the above proposition, is sometimes called a *pair-network* for X . We will continue to use the notation suggested in Proposition 2.1 where if $P \in \mathcal{P}$ then P' denotes the first element of the pair and P'' denotes the second element.

2.2 THEOREM. *Suppose X_1 and X_2 are regular quasi-developable spaces on the same set X and S is the free topological sum of X_1 and X_2 . If the canonical quotient map $\phi: S \rightarrow X$ is pseudo-open, then X is quasi-developable.*

PROOF. Suppose $\mathcal{G} = \bigcup_{n=1}^\infty \mathcal{G}_n$ and $\mathcal{H} = \bigcup_{n=1}^\infty \mathcal{H}_n$ are pair-networks for X_1 , X_2 respectively, where, with the obvious changes, $\mathcal{G}$ and $\mathcal{H}$ satisfy (a) and (b) of Proposition 2.1. For fixed $n, k \in N$, define

$$\mathfrak{B}(n) = \{\text{int } \phi(G''): G \in \mathcal{G}_n\}, \quad \mathfrak{D}(k) = \{\text{int } \phi(H''): H \in \mathcal{H}_k\}.$$

For $G \in \mathcal{G}_n$, $H \in \mathcal{H}_k$, let

$$F(G, H) = \phi(G') \cap \phi(H') - [(\cup \mathcal{B}(n)) \cup (\cup \mathcal{D}(k))]$$

and

$$\mathcal{F}(n, k) = \{F(G, H): G \in \mathcal{G}_n, H \in \mathcal{H}_k\}.$$

Now we show $\mathcal{F}(h, k)$ is a relatively discrete collection—that is, each member of $\mathcal{F}(n, k)$ is contained in an open set which does not intersect any other member of $\mathcal{F}(n, k)$. To see this, pick $F(G, H) \in \mathcal{F}(n, k)$ with $F(G, H) \neq \emptyset$. Note that $F(G, H) \subset \text{int } \phi(G'' \cup H'')$ and $\text{int } \phi(G'' \cup H'') \cap F(G_1, H_1) = \emptyset$ if $G_1 \in \mathcal{G}_n$, $H_1 \in \mathcal{H}_k$ with $G_1 \neq G$ and $H_1 \neq H$. Because of this it suffices to show that

$$F(G, H) \cap \text{cl}(\cup \{F(G, H_1): H_1 \in \mathcal{H}_k, H_1 \neq H\}) = \emptyset \quad (a)$$

and

$$F(G, H) \cap \text{cl}(\cup \{F(G_1, H): G_1 \in \mathcal{G}_n, G_1 \neq G\}) = \emptyset. \quad (b)$$

Suppose $z \in F(G, H) \cap \text{cl}(\cup \{F(G, H_1): H_1 \in \mathcal{H}_k, H_1 \neq H\})$. Let z_1, z_2 be the preimages of z in X_1, X_2 respectively. Now $z_2 \in H'$, so there is an X_2 neighborhood U_2 of z_2 such that $\bar{U}_2 \cap H'_1 = \emptyset$ (closure in X_2) for every $H_1 \in \mathcal{H}_k$, $H_1 \neq H$ (use regularity and the fact that $H'' \cap H'_1 = \emptyset$, if $H \neq H_1$). Clearly $\phi^{-1}(z) = \{z_1, z_2\} \subset G'' \cup U_2$; since $G'' \cup U_2$ is open in S and ϕ is a pseudo-open map we have $z \in \text{int } \phi(G'' \cup U_2)$. There exists $y \in (\text{int } \phi(G'' \cup U_2)) \cap F(G, H_1)$ for some $H_1 \in \mathcal{H}_k$, $H_1 \neq H$. Let y_1, y_2 be the preimages of y in X_1, X_2 respectively; then $y_2 \in H'_1$ and $\bar{U}_2 \cap H'_1 = \emptyset$ so there is a neighborhood W of y_2 in X_2 such that $W \cap \bar{U}_2 = \emptyset$ and $\phi(W) \subset \text{int } \phi(G'' \cup U_2)$. Using the fact that W and U_2 are subsets of X_2 with $W \cap U_2 = \emptyset$ and $\phi(W) \subset \text{int } \phi(G'' \cup U_2)$ we see that $\phi(G'' \cup W) = \phi(G'')$. Since $y_1 \in G' \subset G''$ and $y_2 \in W$ it follows that $y \in \text{int } \phi(G'' \cup W)$. Thus $y \in \text{int}(G'') \in \mathcal{B}(n)$ and $y \in F(G, H_1)$ which contradicts the definition of $F(G, H_1)$. It follows that (a) is true and a similar argument will show (b). Hence $\mathcal{F}(n, k)$ is relatively discrete.

For each $(G, H) \in \mathcal{G}_n \times \mathcal{H}_k$ let $E(G, H)$ be an open neighborhood of $F(G, H)$ such that $E(G, H) \subset \text{int } \phi(G'' \cup H'')$ and $E(G, H) \cap F(G_1, H_1) = \emptyset$ if $(G, H) \neq (G_1, H_1)$. Let $\mathcal{E}(n, k) = \{E(G, H): G \in \mathcal{G}_n, H \in \mathcal{H}_k\}$. To see that

$$\{\mathcal{B}(i)\}_i \cup \{\mathcal{D}(j)\}_j \cup \{\mathcal{E}(n, k)\}_{n,k}$$

is a quasi-development for X let U be an open set in X and let $x \in U$. If x_1 and x_2 are the preimages of x in X_1 and X_2 , respectively, then there are integers n, k with $G \in \mathcal{G}_n$ and $H \in \mathcal{H}_k$ such that

$$x_1 \in G' \subset G'' \subset \phi^{-1}(U) \quad \text{and} \quad x_2 \in H' \subset H'' \subset \phi^{-1}(U).$$

If $x \in \text{int } \phi(G'')$ then $\text{int } \phi(G'')$ is the only element of $\mathcal{B}(n)$ containing x and $x \in \text{St}(x, \mathcal{B}(n)) \subset U$. Similarly, if $x \in \text{int } \phi(H'')$, we have $x \in \text{St}(x, \mathcal{D}(k)) \subset U$. Otherwise, $x \in F(G, H)$ and we obtain

$$x \in E(G, H) = \text{St}(x, \mathcal{E}(n, k)) \subset U.$$

That concludes the proof of the theorem.

The next result, which follows very quickly as a corollary to Theorem 2.2, answers Question 2 posed by Arhangel'skiĭ in [A].

2.3 COROLLARY. *Suppose X_1 and X_2 are regular developable spaces on the same set X . If the canonical quotient map $\phi: S \rightarrow X$ is pseudo-open, then X is developable.*

PROOF. X_1 and X_2 are regular quasi-developable spaces so Theorem 2.2 shows that X is quasi-developable. S is developable, hence perfect, and it is easy to show that the property "closed sets are G_δ sets" is preserved under any pseudo-open finite-to-one map. Thus X is a perfect quasi-developable space and therefore is developable.

The condition, in Theorem 2.2 and Corollary 2.3, that both of the spaces X_1 and X_2 be regular, is crucial as the next example shows.

2.4 EXAMPLE. There are developable spaces X_1 and X_2 with X_1 regular and X_2 Hausdorff such that $\phi: S \rightarrow X$ is pseudo-open but X is not developable (or even quasi-developable).

PROOF. Let $X_1 = X_2 = \{(x, y) \in R \times R: y \geq 0\}$ and let $A = \{(x, y) \in R \times R: y = 0\}$. Describe a topology on X_1 as follows: Points in $X_1 - A$ are isolated and points $(a, 0) \in A$ have a neighborhood base consisting of sets of the form $\{(x, y) \in X_1: y = 0, |x - a| < 1/n\}$, for $n \in N$. Then X_1 is a regular developable space. Describe a topology on X_2 by letting points in $X_2 - A$ be isolated and points $(a, 0) \in A$ have a neighborhood base consisting of sets of the form

$$\{(a, 0)\} \cup \{(x, y) \in X_2: |x - a| < i/n, \\ \text{and } 0 < y < x - a \text{ or } 0 < y < a - x\}.$$

It is easy to see that X_2 is a Hausdorff developable space and the quotient map $\phi: S \rightarrow X$ is pseudo-open but the space X is not developable. In fact, X is semimetrizable, hence perfect and cannot be quasi-developable.

3. Other examples. We conclude with three examples. The first two, Examples 3.1 and 3.2, give negative answers to questions 3 and 4 by Arhangel'skiĭ in [A].

3.1 EXAMPLE. There are completely regular spaces X_1 and X_2 , each with a σ -disjoint base, such that $\phi: S \rightarrow X$ is pseudo-open but X does not have a σ -disjoint base.

PROOF. Let $X_1 = X_2 = \{(x, y) \in R \times R: y \geq 0\}$ and let $A = \{(x, y) \in R \times R: y = 0\}$. Describe a topology on X_1 as follows: Points in $X_1 - A$ are isolated and points $(a, 0) \in A$ have a neighborhood base consisting of sets of the form $\{(x, y) \in X_1: y < 1/n \text{ and } y = x - a\}$, for $n \in N$. Describe a topology on X_2 by letting points in $X_2 - A$ be isolated and points $(a, 0) \in A$

have a neighborhood base consisting of sets of the form $\{(x, y) \in X_2: y < 1/n \text{ and } y = -x + a\}$, for $n \in N$. Then X_1 and X_2 each have a σ -disjoint base (in fact, they are metrizable) and $\phi: S \rightarrow X$ is pseudo-open but the quotient space X does not have a σ -disjoint base. To see this, note that X is homeomorphic to the space of Example 1 of [H] and Heath has shown that this example is a nonscreenable Moore space, and hence could not have a σ -disjoint base.

For the next example, recall that a subset B of the real numbers is called a Q -set if every subset of B is a relative F_σ set. It is known that the existence of uncountable Q -sets is consistent with ZFC. See [T] for further details and a discussion of the history behind Q -sets.

3.2 EXAMPLE (assuming the existence of an uncountable Q -set). There are metrizable spaces X_1 and X_2 such that $\phi: S \rightarrow X$ is pseudo-open and X is normal and first countable but not metrizable.

PROOF. If B is an uncountable Q -set, let $A = B \times \{0\}$ and let $X_1 = X_2 = A \cup \{(x, y) \in R \times R: y > 0\}$. Construct topologies on X_1 and X_2 in the manner of Example 3.1. Then X_1 and X_2 are metrizable, and the quotient space X is a normal metacompact nonmetrizable Moore space as shown by Heath in [H, Theorem 3].

That the quotient space X in Example 3.1 (or 3.2) is metacompact is not surprising in view of a recent result by Arhangel'skii. He solved an old problem by showing that if $f: Y \rightarrow Z$ is a pseudo-open compact mapping from a metric space Y onto a space Z then Z is metacompact [A]. Z was known to be developable, and Arhangel'skii used the subparacompactness of Z along with the collectionwise normality of Y to prove that Z is metacompact. In fact, with hindsight, it is relatively easy to show that if $f: Y \rightarrow Z$ is a pseudo-open compact mapping from a paracompact space Y onto a subparacompact space then Z is metacompact. H. Junnila has recently announced [J] the solution of a more difficult problem by showing that the pseudo-open compact image of a paracompact space is metacompact (without assuming the range is subparacompact). The next example shows that the range of such a mapping need not be subparacompact so that Arhangel'skii's approach to his problem could not be used to prove Junnila's result.

3.3 EXAMPLE. There are locally compact paracompact spaces X_1 and X_2 such that $\phi: S \rightarrow X$ is pseudo-open but X is not subparacompact.

PROOF. Let $X_1 = X_2 = [0, \omega_2) \times [0, \omega_2)$ and for $0 < \alpha < \omega_2$ let

$$H_\alpha = [0, \omega_2) \times \{\alpha\} \quad \text{and} \quad V_\alpha = \{\alpha\} \times [0, \omega_2).$$

Describe a topology on X_1 as follows: If $0 < \alpha < \omega_2$ then neighborhoods of the point $(0, \alpha)$ must contain $(0, \alpha)$ and all but finitely many points of H_α . All other points in X_1 are isolated. Describe a topology on X_2 as follows: If $0 < \alpha < \omega_2$ then neighborhoods of $(\alpha, 0)$ must contain $(\alpha, 0)$ and all but finitely many points of V_α . All other points in X_2 are isolated. It follows that

X_1 and X_2 are locally compact paracompact spaces and $\phi: S \rightarrow X$ is pseudo-open. The image space X is the same as the space in Example 4.1 in [Bu] and it is shown there that X is not subparacompact.

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