EXTENDING MONOTONE DECOMPOSITIONS OF MANIFOLDS

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ABSTRACT. Let M^m ($m \geq 3$) be a compact, connected PL manifold and let $X \subseteq M$ be a proper, closed subset of the interior of M such that for each open, connected subset $U \subseteq M$ either $U - (X \cap U)$ is connected or $X \cap \text{bd}(U) \neq \emptyset$. Let P be a connected and simply connected polyhedron with $\dim P \geq 3$. There exists a monotone mapping f from M onto P with each component of X being a point-inverse of f . In the case with M oriented and P the m -sphere, there exists such a monotone mapping of each degree.

1. Introduction. Bing [Bi; $m = 3$], Bean [Be; $m = 3$], and Coram [Co; $m \geq 4$] showed that if X is a proper, closed subset of a PL manifold M^m such that for each open, connected subset $U \subseteq M$ either $U - (X \cap U)$ is connected or $X \cap \text{bd}(U) \neq \emptyset$, then there is a monotone mapping f from M onto the m -sphere with each component of X being a point-inverse of f . These results are generalized in two ways. First, if P is a simply connected polyhedron with $\dim P \geq 3$, then there is a monotone mapping f from M onto P with each component of X being a point-inverse of f . Second, if M is oriented, then there is such a monotone mapping onto the m -sphere of each degree. The mappings constructed by Bing, Bean and Coram have degree ± 1 ; PL monotone mappings between manifolds of the same dimension have degree ± 1 and their mappings are "almost" PL.

In general, the assumption that P be simply connected cannot be omitted since monotone mappings between ANR's induce surjections between fundamental groups. Adopting a point of view used in [Wa-1], [Wa-2], a generalization to the nonsimply connected case is possible; such a generalization will appear in a subsequent paper.

Since there are limitations placed on those compacta which are monotone images of 2-dimensional manifolds [Ho], [Mo], [R-S], [Yo], the assumption that M^m have dimension at least three is essential. However, the first result holds for certain simply connected 2-dimensional polyhedra and, if X is connected, for simply connected 1-dimensional polyhedra.

The main tool used in the proofs of the above results is an extension theorem in [Wa-3]; roughly, the theorem says that if a mapping from the boundary of an m -manifold ($m \geq 3$) to a simply connected ANR has an

Received by the editors February 8, 1978 and, in revised form, May 30, 1978.

AMS (MOS) subject classifications (1970). Primary 57A60; Secondary 54B15, 54C10, 57A15.

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 0002-9939/79/0000-0185/\$02.25

extension to the entire manifold, then it has an extension which is monotone.

2. Definitions and basic concepts. Spaces are assumed to be separable and metric; we use ∂M to denote the boundary of a manifold and $\text{bd}()$, $\text{int}()$, $\text{cl}()$, and $\text{diam}()$ to denote the topological boundary, topological interior, closure, and diameter, respectively. A *mapping* is a continuous function; a mapping is *monotone* ($= UV^\circ$) if each point-inverse is compact and connected. Let $\mathcal{Q}$ be a collection of sets; $\mathcal{Q}^*$ denotes the union of the elements of $\mathcal{Q}$.

Let M^m ($m \geq 3$) be a compact, connected PL manifold and let $X \subseteq (M - \partial M)$ be a proper, closed subset of M such that for each open connected subset $U \subseteq M$ either $U - (U \cap X)$ is connected or $\text{bd}(U) \cap X \neq \emptyset$. Then there is a sequence $\{\mathfrak{M}_i\}_{i=1}^\infty$ satisfying: (1) $\mathfrak{M}_i = \{M_i^1, \dots, M_i^{q_i}\}$ is a finite collection of pairwise disjoint compact m -dimensional PL submanifolds of $M - \partial M$ with connected boundaries; (2) each M_i^j is contained in the interior of some M_{i-1}^k ; (3) $\bigcap_{i=1}^\infty \mathfrak{M}_i^* = X$. The sequence $\{\mathfrak{M}_i\}_{i=1}^\infty$ is called a *special defining sequence* for X .

The next result is an immediate corollary to [Wa-3; Main Theorem] and is the “key” result needed in this paper.

2.1. THEOREM. *Let M^m ($m \geq 3$) be a compact, connected PL manifold and let P be a compact, simply connected polyhedron. Each mapping from M to P is homotopic rel ∂M to a monotone mapping from M onto P .*

3. Basic extension theorem. This section contains the proof of the following result on extending monotone decompositions; the proofs of subsequent results rely heavily on the proof.

3.1. THEOREM. *Let M^m ($m \geq 3$) be a compact, connected PL manifold, let P be a compact, simply connected polyhedron with $\dim P \geq 3$, and let $X \subseteq (M - \partial M)$ be a proper, closed subset such that for each open connected subset $U \subseteq M$ either $U - (U \cap X)$ is connected or $\text{bd}(U) \cap X \neq \emptyset$. There is a monotone mapping f from M onto P with each component of X being a point-inverse of f .*

PROOF. Let $\mathfrak{M}_i = \{M_i^1, \dots, M_i^{q_i}\}$, $i = 1, 2, \dots$, be a special defining sequence for X with $\mathfrak{M}_1 = \{M_1^1\}$. Let $n = \dim P$ and let $B \subseteq P$ be a PL n -ball. For $i = 1, 2, \dots$, let $\mathfrak{B}_i = \{B_i^1, \dots, B_i^{q_i}\}$ be a collection of pairwise disjoint PL n -balls with diameters less than $1/i$ such that $B_1^1 \subseteq (B - \partial B)$ and $B_i^j \subseteq (B_{i-1}^k - \partial B_{i-1}^k)$ if and only if $M \subseteq (M_{i-1}^k - \partial M_{i-1}^k)$.

Since P is simply connected and $n \geq 3$, each of the sets $\text{cl}(P - B_1^1)$ and $\text{cl}(B_i^j - \mathfrak{B}_{i+1}^*)$ is simply connected, and since the $\mathfrak{M}_i$'s form a special defining sequence, each of the sets $\text{cl}(M - M_1^1)$ and $\text{cl}(M_i^j - \mathfrak{M}_{i+1}^*)$ is connected. Let f' be a mapping from $\bigcup \{\partial M_i^j: i \geq 1 \text{ and } 1 \leq j \leq q_i\}$ to $\bigcup \{\partial B_i^j: i \geq 1 \text{ and } 1 \leq j \leq q_i\}$ such that f' maps ∂M_i^j onto ∂B_i^j and is null-homotopic (as a mapping from ∂M_i^j to ∂B_i^j). The mapping f' from ∂M_i^j to

∂B_i^j can be defined by letting $f' = \alpha \circ \beta$ where β maps ∂M_i^j onto the unit interval I and α maps I onto ∂B_i^j .

Since f' is null-homotopic, f' extends to a mapping from $\text{cl}(M - M_1^1)$ to $\text{cl}(P - B_1^1)$ and, therefore, Theorem 2.1 implies that f' extends to a monotone mapping from $\text{cl}(M - M_1^1)$ onto $\text{cl}(P - B_1^1)$. Similarly, f' extends to a monotone mapping from $\text{cl}(M_i^j - \mathfrak{N}_{i+1}^*)$ onto $\text{cl}(B_i^j - \mathfrak{B}_{i+1}^*)$. Finally, extend f' to a mapping f from M onto P by letting $f(\cap_{i=1}^\infty M_i^j) = \cap_{i=1}^\infty B_i^j$ for each nested collection $M_1^j \supseteq M_2^j \supseteq \dots$. The mapping f is the desired monotone mapping from M onto P .

4. Extensions of different degrees. In this section, assuming that M^m is orientable and without boundary and that P is the m -sphere, it is shown that there is a monotone mapping f as in Theorem 3.1 of each degree. The monotone mapping constructed in §3 is easily seen to have degree zero.

4.1. THEOREM. *Let M^m ($m \geq 3$) be a compact, connected oriented PL manifold without boundary and let $X \subseteq M$ be a proper closed subset such that for each open connected subset $U \subseteq M$ either $U - (U \cap X)$ is connected or $X \cap \text{bd}(U) \neq \emptyset$. There exists a monotone mapping f of each degree from M onto the m -sphere S^m with each component of X being a point-inverse of f .*

PROOF. Following the notation from the proof of Theorem 3.1, define f' so that $f': \partial M_i^j \rightarrow \partial B_i^j$ is onto and has degree d . Since $\text{cl}(S^m - B_1^1)$ is an m -ball, f' extends to a mapping from $\text{cl}(M - M_1^1)$ to $\text{cl}(S^m - B_1^1)$; Theorem 2.1 guarantees that the extension can be chosen to be monotone and onto. If f' extends to a mapping from $\text{cl}(M_i^j - \mathfrak{N}_{i+1}^*)$ to $\text{cl}(B_i^j - \mathfrak{B}_{i+1}^*)$, then Theorem 2.1 implies that each such extension can be chosen to be monotone and onto. An argument using the Mayer-Vietoris sequence would show that the induced mapping has degree d . Lemma 4.3 below guarantees that the necessary extensions of f' exist.

In the following lemmas, it is necessary to make appropriate choices of orientations for the various boundary components.

4.2. LEMMA. *Let $B = S^m - \bigcup_{i=1}^n \text{int}(B_i)$ where the B_i 's are pairwise disjoint PL m -balls and let d be an integer. If $m \geq 3$, then there is a mapping $f: (B; \partial B_1, \dots, \partial B_n) \rightarrow (B; \partial B_1, \dots, \partial B_n)$ with $f: \partial B_i \rightarrow \partial B_i$ having degree d for each $i = 1, \dots, n$.*

PROOF. We assume that $m = 3$, since the higher dimensional cases are proved in a similar manner. Let $f': S^3 \rightarrow S^3$ be a mapping which is the second suspension of a degree d mapping from S^1 to itself. There are pairwise disjoint PL 3-balls $B'_1, \dots, B'_n$ (centered on the suspension circle) with $f': (B'_i, \partial B'_i) \rightarrow (B'_i, \partial B'_i)$ and with $f': \partial B'_i \rightarrow \partial B'_i$ having degree d . The mapping f is obtained by conjugating f' with an appropriate homeomorphism of S^3 and restricting the mapping to B .

4.3. LEMMA. *Let $B = S^m - \bigcup_{i=1}^n \text{int}(B_i)$ where the B_i 's are pairwise disjoint PL m -balls, let M^m be a compact, connected, oriented PL manifold with the boundary of M having components $\partial M_1, \dots, \partial M_n$, and let d be an integer. If $m \geq 3$, there is a mapping $f: (M; \partial M_1, \dots, \partial M_n) \rightarrow (B; \partial B_1, \dots, \partial B_n)$ with $f: \partial M_i \rightarrow \partial B_i$ having degree d .*

PROOF. It suffices to consider the case with $d = 1$, since the other cases follow by composing with the mappings obtained in Lemma 4.2. Let $T_1, \dots, T_{n-1}$ (respectively, $T'_1, \dots, T'_{n-1}$) be pairwise disjoint tubes—i.e., relative regular neighborhoods of arcs—such that T_i (respectively, T'_i) connects ∂B_i and ∂B_n (respectively, ∂M_i and ∂M_n). Let f' be a mapping from $(\bigcup_{i=1}^n \partial M_i) \cup (\bigcup_{i=1}^{n-1} T_i)$ to $(\bigcup_{i=1}^n \partial B_i) \cup (\bigcup_{i=1}^{n-1} T_i)$ with $f': \partial M_i \rightarrow \partial B_i$ having degree 1 and with $f': T'_i \rightarrow T_i$ a homeomorphism. Since $\text{cl}(B - \bigcup_{i=1}^{n-1} T_i)$ is an n -ball, f' extends to the desired mapping f .

5. One and two dimensional cases. The following are versions of Theorem 3.1 which hold when P is a 1- or 2-dimensional polyhedron.

5.1. THEOREM. *Let M^m ($m \geq 3$) and X be as in Theorem 3.1 and let P be a compact, simply connected 2-dimensional polyhedron such that when P is triangulated some 2-simplex has a free face. There is a monotone mapping f from M onto P with each component of X being a point-inverse of f .*

PROOF. Let $B \subseteq P$ be a 2-simplex (in some triangulation of P) with ∂B containing an arc A which is a free face. Following the notation in the proof of Theorem 3.1, choose the $\mathcal{B}_i$'s such that $B_1^1 \subseteq B - \text{bd}(B)$, $B_i^j \subseteq (B_{i-1}^k - \text{bd}(B_{i-1}^k))$ if and only if $M_i^j \subseteq (M_{i-1}^k - \partial M_{i-1}^k)$, and $\partial B_i^j \cap A$ is an arc. The important feature is that each of the sets $\text{cl}(P - B)$ and $\text{cl}(B_i^j - \mathcal{B}_{i+1}^*)$ is simply connected.

5.2. THEOREM. *Let M^m ($m \geq 3$) and X be as in Theorem 3.1 with the additional assumption that X be connected and let P be a compact, simply connected 1-dimensional polyhedron. There is a monotone mapping f from M onto P with X being a point-inverse of f .*

PROOF. The polyhedron P is necessarily a finite tree, the proof of Theorem 3.1 applies provided the $\mathcal{B}_i$'s are chosen so that $\cap \mathcal{B}_i^*$ is an endpoint of P .

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