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ON THE PROBLEM OF PARTITIONING $\{1, 2, \ldots, n\}$ INTO SUBSETS HAVING EQUAL SUMS

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ABSTRACT. Let N denote the set of natural numbers and let $Z_n = \{1, 2, \ldots, n\}$. For S a finite subset of N, let σS denote the sum of the elements in S. Then $\sigma Z_n = n(n+1)/2$. Suppose $n(n+1) = 2st$, where s and t are integers and $t \geqslant n$. We show that Z_n can be partitioned into $T_1 \cup T_2 \cup \ldots \cup T_s$ such that $\sigma T_i = t$, for $1 \leqslant i \leqslant s$. Such a partition is called an (s, t)-partition of Z_n.

A graph G having $n(n+1)/2$ edges is said to be path-perfect if the edge set of G can be partitioned as $E_1 \cup E_2 \cup \ldots \cup E_n$ so that E_i induces a path of length i, for $1 \leqslant i \leqslant n$. Suppose p and n are positive integers and r is an even positive integer with $p \geqslant r + 1$ and $pr = n(n+1)$. The existence of an $(r/2, p)$-partition of Z_n is used to show the existence of an r-regular path-perfect graph G having p vertices and $n(n+1)/2$ edges.

1. Introduction. Let N denote the set of natural numbers and let $Z_n = \{1, 2, \ldots, n\}$. For S a finite subset of N, let σS denote the sum of the elements belonging to S.

It is well known that $\sigma Z_n = n(n+1)/2$. We consider the following question. Suppose $n(n+1) = 2st$, where s and t are integers and $t \geqslant n$. Do there exist s pairwise disjoint subsets of Z_n, say $T_1, T_2, \ldots, T_s$, such that $Z_n = T_1 \cup T_2 \cup \ldots \cup T_s$ and $\sigma T_i = t$, for each i, $1 \leqslant i \leqslant s$? If so, then the partition $T_1 \cup T_2 \cup \ldots \cup T_s$ will be called an *(s, t)-partition* of Z_n. As an example, let $n = 20$, $s = 6$, and $t = 35$. The subsets $T_1 = \{15, 20\}$, $T_2 = \{16, 19\}$, $T_3 = \{17, 18\}$, $T_4 = \{4, 8, 9, 14\}$, $T_5 = \{5, 7, 10, 13\}$, and $T_6 = \{1, 2, 3, 6, 11, 12\}$ form a $(6, 35)$-partition of Z_{20}.

This number theoretical problem arose in the context of a problem in graph theory. In [3], Straight considered the question of the existence of regular path-perfect graphs, a concept to be explained in §3. While doing so, he was able to find a partial solution to the problem concerning (s, t)-partitions of Z_n. He posed the problem during a talk at SUNY-Fredonia, and from there it was communicated to Schillo, who found a complete solution.

In §2 we show that an (s, t)-partition of Z_n exists whenever $n(n+1) = 2st$. In §3 this result is then applied to the question of the existence of even-regular, path-perfect graphs.

2. The main result.

THEOREM 1. *Let n be a positive integer and suppose $n(n+1) = 2st$, where s and t are integers and $t \geqslant n$. Then there exists an (s, t)-partition of the set Z_n.*

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PROOF. We employ induction on n. It is easy to verify the theorem for small values of n. Now let n be some fixed positive integer and suppose the theorem is true whenever $\hat{n}(\hat{n}+1) = 2\hat{s}\hat{t}$, $\hat{t} \geqslant \hat{n}$, and $\hat{n} < n$. Let s and t be given with $n(n+1) = 2st$ and $t \geqslant n$. We wish to find sets $T_1, T_2, \ldots, T_s$ forming an (s, t)-partition of Z_n.

Case 1A. Suppose $n \leqslant t < 2n$ and t is even. Let $m = (2n - t)/2$ and for $1 \leqslant i \leqslant m$, let $T_i = \{n - i + 1, t - n + i - 1\}$. Note that the elements of Z_n not belonging to $T_1 \cup T_2 \cup \ldots \cup T_m$ form the set $Z_{t-n-1} \cup \{t/2\}$. Now $\sigma Z_{t-n-1} = (t/2)(t - 2n + 2s - 1)$ and $t - n - 1 < n$. Thus, by the induction hypothesis, there exists a $(t - 2n + 2s - 1, t/2)$-partiton of Z_{t-n-1}. Combine one of the sets in this partition with $\{t/2\}$ to obtain T_{m+1}, and unite the rest two at a time to form $T_{m+2}, \ldots, T_s$. Then $T_1 \cup T_2 \cup \ldots \cup T_s$ is an (s, t)-partition of Z_n.

Case 1B. Suppose $n \leqslant t < 2n$ and t is odd. Let $k = (2n - t + 1)/2$ and for $1 \leqslant i \leqslant k$, let $T_i = \{n - i + 1, t - n + i - 1\}$. After forming $T_1, \ldots, T_k$, the subset of Z_n which remains is Z_{t-n-1}. Now $\sigma Z_{t-n-1} = t((t - 1 - 2n + 2s)/2)$. So by the induction hypothesis there exists a $((t - 1 - 2n + 2s)/2, t)$-partition of Z_{t-n-1}. Letting the elements of this partition be $T_{k+1}, \ldots, T_s$, we obtain an (s, t)-partition of Z_n.

Case 2. Suppose $t \geqslant 2n$. Using the induction hypothesis, let $T_1' \cup T_2' \cup \ldots \cup T_s'$ be an $(s, t - 2n + 2s - 1)$-partition of Z_{n-2s}. Next let $T_i = T_i' \cup \{n - 2s + i, n - i + 1\}$ for $1 \leqslant i \leqslant s$. Then $T_1 \cup T_2 \cup \ldots \cup T_s$ is an (s, t)-partition of Z_n.

Therefore, by induction, the theorem is proven. □

3. An application to graph theory. Let K_n and P_n denote the complete graph of order n and the path of length n, respectively. A *factorization* of a graph G is a partition of its edge set. The subgraphs of G induced by the elements in the partition are called *factors*. In [1], Fink and Straight consider the problem of factoring a graph into paths of different lengths. Specifically, they define a graph G having size $n(n+1)/2$ to be *path-perfect* if the edge set of G can be partitioned as $E_1 \cup E_2 \cup \ldots \cup E_n$ so that E_i induces P_i, for $1 \leqslant i \leqslant n$. For example, the Petersen graph, shown in Figure 1, is path-perfect. To see this, let $P_1 = v_7v_8$, $P_2 = v_4v_9v_{10}$, $P_3 = v_5v_1v_8v_9$, $P_4 = v_3v_7v_6v_{10}v_2$ and $P_5 = v_1v_2v_3v_4v_5v_6$.

Note also that the Petersen graph is 3-regular (each vertex is incident with 3 edges). If G is an r-regular graph of order p and size $n(n+1)/2$, then

$$pr = n(n+1). \tag{1}$$

Thus r must divide $n(n+1)$. Considering the case where r divides n or $n+1$, Straight [3] showed the following.

THEOREM A. *Let r be an odd positive integer. There exists an r-regular path-perfect graph of order $m(mr+1)$ if, and only if, m equals* 1. *Also, there exists an r-regular path-perfect graph of order $m(mr-1)$ if, and only if, m equals* 2.

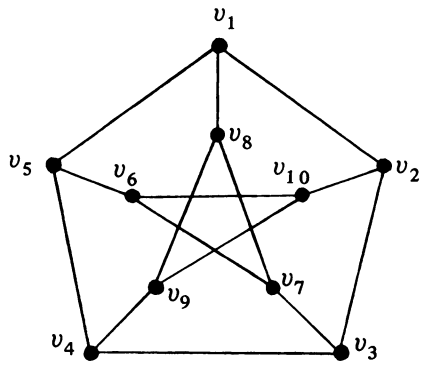

FIGURE 1

If r is even, then equation (1) can be written as $p(r/2) = n(n + 1)/2$. We now wish to apply Theorem 1 with $t = p$ and $s = r/2$ to help prove the following.

THEOREM 2. *Let p and n be positive integers and let r be an even positive integer such that $p \geqslant r + 1$ and $pr = n(n + 1)$. Then there exists an r-regular path-perfect graph G of order p and size $n(n + 1)/2$.*

PROOF. Let p, n and r be given satisfying the conditions of the theorem. We shall construct G.

It is a well-known result in graph theory (see [**2**, p. 89]) that if p is even, K_p can be factored into $(p - 2)/2$ hamiltonian cycles and a factor which is 1-regular, while if p is odd, K_p can be factored into $(p - 1)/2$ hamiltonian cycles. In either case, we may combine $r/2$ of these cycles, call them $C_1, C_2, \ldots, C_{r/2}$, to form an r-regular spanning subgraph of K_n. This subgraph will be G.

To show that G is path-perfect, apply Theorem 1 with $s = r/2$ and $t = p$. For $1 \leqslant i \leqslant s$, the subset T_i of Z_n given by the theorem tells us how to partition C_i into paths; that is, the elements of T_i are the lengths of the paths. □

Theorem 2 answers the question of the existence of r-regular path-perfect graphs when r is even. Theorem A can be applied when r is odd and r divides n or r divides $n + 1$. In particular, one can apply Theorem A whenever r is a power of some odd prime. This leaves one unsolved case–when r is odd and r divides neither n nor $n + 1$. For example, we do not know whether there exists a 15-regular path-perfect graph of order 28 and size $20(21)/2 = 210$.

REFERENCES

1. J. F. Fink and H. J. Straight, *Path-perfect graphs*, Discrete Math. (to appear).

2. F. Harary, *Graph theory*, Addison-Wesley, Reading, Mass., 1971.

3. H. J. Straight, *Partitions of the vertex set or edge set of a graph*, Doctoral Dissertation, Western Michigan University, 1977.

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