GENERALIZED PARABOLIC FUNCTIONS USING THE PERRON-WIENER-BRELOT METHOD

NEIL EKLUND

ABSTRACT. Let L be a linear, second order parabolic operator in divergence form and let U be a bounded domain in E^{n+1} . The Dirichlet problem for $Lu = 0$ is solved in U using the Perron-Wiener-Brelot method.

In a preceding paper [1] the author has obtained existence and uniqueness of solutions for a class of parabolic operators of second order defined in a bounded cylinder in E^{n+1} . In this paper existence and uniqueness are obtained in a bounded domain in E^{n+1} using subsolutions.

The operator under consideration is defined by

$$Lu = u_t - [a_{ij}u_{,i} + d_j u]_{,j} - b_j u_{,j} - cu$$

where the coefficients are measurable functions defined on $U \subset \Omega \times (0, T)$ for $\Omega \subset E^n$ bounded and finite T and

(i) $|a_{ij}| \leq M$ almost everywhere on U and, for some $\lambda > 0$, $a_{ij}(x, t)z_i z_j \geq \lambda|z|^2$ for all $z \in E^n$ and almost all $(x, t) \in U$.

(ii) $c(x, t) \in L^{p,q}(U)$ for some pair p, q satisfying

$$1 < p, q \leq \infty, \quad n/2p + 1/q < 1. \quad (*)$$

(iii) $b_j(x, t), d_j(x, t) \in L^{p,q}(U)$ for each j for some pair p, q satisfying

$$2 < p, q \leq \infty, \quad n/2p + 1/q < 1/2. \quad (**)$$

(iv) $c + [d_j]_{,j} \leq 0$ weakly in U .

Here $(x, t) = (x_1, x_2, \dots, x_n, t) \in E^{n+1}$, $u_{,j} = \partial u / \partial x_j$, and repetition of i and/or j means summation over such indices for $1 \leq i, j \leq n$.

To increase readability, the following definitions and theorems from the references are included:

DEFINITION A. If $Q = \Omega \times (0, T]$ is a cylindrical set in E^{n+1} , then the *parabolic boundary* of Q is given by

$$\partial_p Q = \{\partial\Omega \times [0, T]\} \cup \{\Omega \times (0)\}.$$

DEFINITION B. Let Q be a bounded cylindrical set in E^{n+1} and let u be given on $\partial_p Q$. Then $v(z) = L(u; z, Q)$ is the *solution* of the boundary value problem $Lv = 0$ in Q , $v = u$ on $\partial_p Q$.

Received by the editors December 2, 1976 and, in revised form, June 20, 1978.

AMS (MOS) subject classifications (1970). Primary 35K20; Secondary 35D05.

Key words and phrases. Parabolic PDE, Perron-Wiener-Brelot solution, subsolutions.ams78

© 1979 American Mathematical Society
0002-9939/79/0000-0209/\$02.75

DEFINITION C (DEFINITION 4 OF [2]). Let U be a bounded, open set in E^{n+1} . u is *locally parabolic* at z if there is a neighborhood of z , N_z , such that $u(z) = L(u; z, R)$ for all rectangles R , $\bar{R} \subset N_z$.

DEFINITION D (DEFINITION 6 OF [2]). u is *superparabolic* ($\in \mathcal{S}'_U$) on U if

- (i) $u \not\equiv +\infty$ on U ,
- (ii) $u > -\infty$ on U ,
- (iii) u is lower semicontinuous on U ,
- (iv) for each $z \in U$ and each rectangle R with $z \in R$ and $\bar{R} \subset U$, $L(u; z, R) \leq u(z)$ on R .

DEFINITION E (DEFINITION 7 OF [3]). Let $z = (x, t)$, $w = (y, s) \in U$. $z < w$ in U if there is a polygonal path $C_{z,w}(\alpha)$, $0 \leq \alpha \leq 1$, such that

- (i) $C_{z,w}(0) = z$, $C_{z,w}(1) = w$,
- (ii) if $C_{z,w}(\alpha) = (x_\alpha, t_\alpha)$, then $\alpha < \beta$ implies $t_\alpha \leq t_\beta$,
- (iii) $\{C_{z,w}(\alpha); 0 \leq \alpha \leq 1\} \subset U$.

DEFINITION F (DEFINITION 8 OF [3]). Let u be defined on U , $U_p = \{z \in U; u \text{ locally parabolic at } z\}$, and $U_\pm = \{z \in U; u(z) = \pm \infty\}$. u is said to have property P_+ (or P_-) if

- (i) $w \in U_p$ implies $z \in U_p$ for all $z \in U$ with $z < w$,
- (ii) $w \in U_+$ (or U_-) implies $z \in U_+$ (or U_-) for all $z \in U$ with $z > w$,
- (iii) $U - U_p - U_+ - U_-$ is a finite union of sets of the form $\Omega_j \times (t_j)$.

DEFINITION G (DEFINITION 10 OF [3]). A family $\mathcal{F}$ in $\mathcal{S}'_U$ is *saturated* if

- (i) $u, v \in \mathcal{F} \Rightarrow u \wedge v \in \mathcal{F}$,
- (ii) $u \in \mathcal{F} \Rightarrow u^* \in \mathcal{F}$ where, for some standard rectangle R in U ,

$$u^*(z) = \begin{cases} u(z) & \text{in } U - R, \\ L(u; z, R) & \text{in } R. \end{cases}$$

THEOREM A (THEOREM 7 IN [2]). If $u, v \in \mathcal{S}'_U$ then $u \wedge v \in \mathcal{S}'_U$.

THEOREM B (THEOREM 4' OF [3]). If $u \in \mathcal{S}'_U$ and if there is a $z_0 \in U$ such that $0 > u(z_0) = \inf_U u$, then $u(z) = u(z_0)$ for all $z < z_0$ in U .

THEOREM C (THEOREM 12 OF [3]). If $\{u_j\}$ is an increasing (or decreasing) sequence of parabolic functions on U , then $u(z) = \lim u_j(z)$ satisfies property P_+ (or P_-).

THEOREM D (THEOREM 13 OF [3]). If $\mathcal{F}$ is a saturated family in $\mathcal{S}'_U$, then $v = \inf_{\mathcal{F}} u$ satisfies property P_- in U .

DEFINITION 1. The extended real-valued function u is *hyperparabolic* (or *hypoparabolic*) on U if it is superparabolic (or subparabolic) or identically $+\infty$ (or $-\infty$) on U .

In Definition E the notation $z < w$ was introduced to mean that it was possible to go from z to w by a polygonal path lying entirely in U with t increasing. The notation $\lim_{z \rightarrow y} u(z)$ will mean the limit of $u(z)$ as z approaches y with $z > y$. This notation will also be used with $\liminf$ and $\limsup$.

DEFINITION 2. Let $P_U = \{z \in \partial U; \text{there is } z_j \rightarrow z \text{ with } z_j \in U \text{ and } z_j > z\}$. Then $\partial_p U = \overline{P_U}$ is called the *parabolic boundary* of U .

DEFINITION 3. The *upper class* of functions $\mathcal{U}_f$ determined by the function f on $\partial_p U$ is given by

$$\mathcal{U}_f = \left\{ u; u \text{ hyperparabolic on } U, \liminf_{z \rightarrow y} u(z) \geq f(y) \right. \\ \left. \text{for all } y \in \partial_p U, u \text{ bounded below on } U \right\}.$$

The *lower class* $\mathcal{L}_f$ determined by f is given by

$$\mathcal{L}_f = \left\{ u; u \text{ hypoparabolic on } U, \limsup_{z \rightarrow y} u(z) \leq f(y) \right. \\ \left. \text{for all } y \in \partial_p U, u \text{ bounded above on } U \right\}.$$

Note that $u \equiv +\infty \in \mathcal{U}_f$ and $u \equiv -\infty \in \mathcal{L}_f$.

DEFINITION 4. Let f be defined on $\partial_p U$. The *upper solution* for the generalized Dirichlet problem is defined to be

$$\overline{P}_f = \inf\{u; u \in \mathcal{U}_f\}.$$

The *lower solution* is defined by

$$P_f = \sup\{u; u \in \mathcal{L}_f\}.$$

LEMMA 1. $\overline{P}_f$ (or P_f) is either identically $+\infty$ (or $-\infty$) or has the property P_- (or P_+) on U .

PROOF. The proof shall be carried out for $\overline{P}_f$ only. If $\mathcal{U}_f = \{+\infty\}$, there is nothing to prove. Suppose $\mathcal{U}_f$ contains a hyperparabolic function which is not identically $+\infty$. Then $\overline{P}_f = \inf\{u \in \mathcal{U}_f; u \in \mathcal{S}'_U\}$. The desired result will follow from Theorem D if $\mathcal{U}_f \cap \mathcal{S}'_U$ can be shown to be saturated in $\mathcal{S}'_U$. Let $u, v \in \mathcal{U}_f \cap \mathcal{S}'_U$. Then, by Theorem A and the definition of $\mathcal{U}_f$, $u \wedge v \in \mathcal{U}_f \cap \mathcal{S}'_U$. Moreover, if $u \in \mathcal{U}_f \cap \mathcal{S}'_U$, then $u^* \in \mathcal{S}'_U$, and, since $u^* = u$ in a neighborhood of ∂U , $u^* \in \mathcal{U}_f$. Therefore, $u^* \in \mathcal{U}_f \cap \mathcal{S}'_U$ and $\mathcal{U}_f \cap \mathcal{S}'_U$ is saturated in $\mathcal{S}'_U$.

DEFINITION 5. If $\overline{P}_f = \overline{P}_f = P_f$ and P_f is a solution of $Lu = 0$ in U , then f is called a *resolutive boundary function* for f .

LEMMA 2. If $u \in \mathcal{L}_f$ and $v \in \mathcal{U}_f$, then $u \leq v$.

PROOF. If either $v = +\infty$ or $u = -\infty$, then $u \leq v$ trivially. Thus, assuming v is superparabolic and u is subparabolic on U , $v - u \in \mathcal{S}'_U$. Then, for $y \in \partial_p U$ with $f(y)$ finite,

$$\liminf_{z \rightarrow y} (v - u)(z) \geq \liminf_{z \rightarrow y} v(z) - \limsup_{z \rightarrow y} u(z) \geq 0.$$

If $y \in \partial_p U$ with $f(y) = +\infty$, then u bounded above and $\liminf_{z \rightarrow y} v(z) = +\infty$ imply that $\liminf_{z \rightarrow y} (v - u)(z) > 0$. Similarly, if $f(y) = -\infty$, then v bounded below and $\limsup_{z \rightarrow y} u(z) = -\infty$ imply $\liminf_{z \rightarrow y} (v - u)(z) > 0$.

It follows from Theorem B and $\liminf_{z \rightarrow y} (v - u)(z) \geq 0$ for all $y \in \partial_p U$ and that $v - u \geq 0$ on U .

COROLLARY. *Let f be defined on $\partial_p U$. Then $\underline{P}_f \leq \bar{P}_f$.*

THEOREM 1. *If f is bounded on $\partial_p U$ and there is a solution u of $Lu = 0$ on U such that $\lim_{z \rightarrow y} u(z) = f(y)$ for all $y \in \partial_p U$, then f is resolvable and $P_f = u$.*

PROOF. Since f is bounded on $\partial_p U$ it follows from Theorem B that u is bounded on U . Hence, $u \in \mathcal{E}_f \cap \mathcal{U}_f$ and $\bar{P}_f \leq u \leq \underline{P}_f$. However, by the Corollary to Lemma 2, $\underline{P}_f \leq \bar{P}_f$. Therefore f is resolvable.

The following lemma gives some elementary properties of lower and upper solutions and its proof is left to the reader.

LEMMA 3. *Let f and g be extended real-valued functions on $\partial_p U$ and let $c \geq 0$ be any constant. Then*

- (i) $\bar{P}_c \leq c$.
- (ii) $\bar{P}_{cf} = c\bar{P}_f$ and $\underline{P}_{cf} = c\underline{P}_f$. If f is resolvable, then cf is resolvable and $P_{cf} = cP_f$.
- (iii) $\bar{P}_{-f} = -\underline{P}_f$. If f is resolvable then $-f$ is also and $P_{-f} = -P_f$.
- (iv) If $f \leq g$, then $\bar{P}_f \leq \bar{P}_g$ and $\underline{P}_f \leq \underline{P}_g$.
- (v) $\bar{P}_{f+g} \leq \bar{P}_f + \bar{P}_g$ and $\underline{P}_{f+g} \geq \underline{P}_f + \underline{P}_g$ wherever the sums are defined. If f and g are resolvable and $f + g$ is defined, then $f + g$ is resolvable and $P_{f+g} = P_f + P_g$.

LEMMA 4. *The real-valued resolvable functions form a linear class. Moreover, if $\{f_i\}$ is a sequence of such functions which converges uniformly to f , then f is resolvable and P_f , the Dirichlet solution corresponding to f , converges uniformly to P_f .*

PROOF. The first assertion follows from parts (ii) and (v) of Lemma 3. To see the other claim, let $\varepsilon > 0$ be arbitrary. Then, for sufficiently large i , $|f - f_i| < \varepsilon$ on $\partial_p U$ and hence $f < f_i + \varepsilon$ and $f_i < f + \varepsilon$. Therefore, $\bar{P}_f < \bar{P}_i + \varepsilon$ and $\bar{P}_i < \bar{P}_f + \varepsilon$ and hence $|\bar{P}_f - \bar{P}_i| < \varepsilon$ on U . Similarly, $|\underline{P}_f - \underline{P}_i| < \varepsilon$ on U . Since each f_i is resolvable, it follows that $|\underline{P}_f - \bar{P}_f| < |\underline{P}_f - \underline{P}_i| + |\bar{P}_f - \bar{P}_i| < 2\varepsilon$ and f is resolvable. The uniform convergence of $\{P_i\}$ to P_f on U follows since $|P_f - P_i| = |\underline{P}_f - \underline{P}_i| < \varepsilon$ on U for sufficiently large i .

THEOREM 2. *If $\{f_i\}$ is an increasing sequence of boundary functions, $f = \lim f_i$, and $\bar{P}_1 > -\infty$ on U , then either $\bar{P}_f(z_0) = \infty$ or $\bar{P}_f(z_0) = \lim \bar{P}_i(z_0)$ for each $z_0 \in U$. If, in addition, each f_i is resolvable, then $\bar{P}_f = \underline{P}_f$ and f is resolvable if either $\bar{P}_f$ or $\underline{P}_f$ is finite.*

PROOF. If $\bar{P}_f \equiv \infty$ on U , there is nothing to prove. Thus, assume there is a $z_0 \in U$ with $\bar{P}_f(z_0) < \infty$. Since $f_i \uparrow f$, $\bar{P}_i(z_0) > -\infty$ implies $-\infty < \bar{P}_i(z_0) \leq \bar{P}_f(z_0) < \infty$. Letting $U_0 = \{z \in U; z < z_0\}$ it follows from Lemma 1 that each $\bar{P}_i$ satisfies property P_- and hence is parabolic on U_0 . It follows from Theorem C that $\lim \bar{P}_i$ is in P_+ and hence parabolic on U_0 .

To see that $\bar{P}_f(z_0) = \lim \bar{P}_i(z_0)$, let $\varepsilon > 0$ be arbitrary. For each j let $v_j \in \mathcal{Q}_j$ be chosen so that

$$v_j(z_0) < \bar{P}_j(z_0) + \varepsilon \cdot 2^{-j}. \quad (1)$$

Then set

$$v = \lim \bar{P}_i + \sum_{j=1}^{\infty} (v_j - \bar{P}_j).$$

Since $\bar{P}_j$ is parabolic on U_0 , (1) and $v_j \in \mathcal{Q}_j$ imply that $v_j - \bar{P}_j$ is superparabolic on U_0 . Moreover, $v_j - \bar{P}_j \geq 0$. Therefore, for each j ,

$$v \geq \lim_i \bar{P}_i + (v_j - \bar{P}_j) = \lim_i (\bar{P}_i - \bar{P}_j) + v_j.$$

However, for each $i > j$ Lemma 3 implies $\bar{P}_i \geq \bar{P}_j$ and hence $v \geq v_j$ for each j . Also, $v_j \in \mathcal{Q}_j$ implies v_j , and hence v , is bounded below.

Now each $v_j - \bar{P}_j$ is superparabolic and hence $\Sigma(v_j - \bar{P}_j)$ is hyperparabolic. But $\Sigma(v_j - \bar{P}_j)(z_0) < \varepsilon$ from (1) and hence $\Sigma(v_j - \bar{P}_j)$ is superparabolic. Recalling that $\lim_i \bar{P}_i$ is parabolic on U_0 it follows that v is superparabolic on U_0 .

For any $y \in \partial_p U_0$,

$$\liminf_{z \rightarrow y} v(z) \geq \liminf_{z \rightarrow y} v_j(z) \geq f(y).$$

Since the left side is independent of j ,

$$\liminf_{z \rightarrow y} v(z) \geq f(y)$$

and $v \in \mathcal{Q}_f$. Therefore, $v(z_0) \geq \bar{P}_f(z_0)$ and hence

$$\lim \bar{P}_j(z_0) \leq \bar{P}_f(z_0) \leq v(z_0) \leq \lim \bar{P}_i(z_0) + \varepsilon.$$

Since ε was arbitrary $\bar{P}_f(z_0) = \lim \bar{P}_i(z_0)$. If each f_i is resolute, then for any z_0 with $\lim \bar{P}_i(z_0) < \infty$

$$\bar{P}_f(z) = \lim \bar{P}_j(z) = \lim \underline{P}_j(z) \leq \underline{P}_f(z) \leq \bar{P}_f(z)$$

for any $z \in U_0$. Hence f is resolute on U_0 .

LEMMA 5. *If u is a bounded subparabolic function on the bounded domain U and $f(y) = \lim_{z \rightarrow y} u(z)$ exists for all $y \in \partial_p U$, then f is a resolute boundary function.*

PROOF. Since f is bounded, $\bar{P}_f$ and $\underline{P}_f$ are bounded and hence parabolic on U . Since $u \in \mathcal{L}_f$, $u \leq \underline{P}_f$ and hence

$$\liminf_{z \rightarrow y} \underline{P}_f(z) \geq \liminf_{z \rightarrow y} u(z) = f(y)$$

for all $y \in \partial_p U$. Therefore $\underline{P}_f \in \mathcal{Q}_f$ and hence $\underline{P}_f \geq \bar{P}_f$. It follows that f is resolute.

In Lemma 6 we make one more restriction on the coefficients:

(v) $|b_j + d_j|$ is essentially bounded on U for each j ; that is, there is a constant M' such that $|b_j + d_j| \leq M'$ on $U - V$ where $\text{meas}(V) = 0$.

LEMMA 6. Let K be any compact subset of U . Let f be any continuous function on K . Then, for any $\varepsilon > 0$, there is a function u which is the difference of two continuous subparabolic functions on a domain containing K such that $\sup_K |u - f| < \varepsilon$.

PROOF. Since f is continuous on the compact set K it follows from the Weierstrass approximation theorem that there is a polynomial u on K such that $\sup_K |u - f| < \varepsilon$. It must be shown that u can be written as a difference of two subparabolic functions. Extend naturally the domain of u to U .

Since $L(t+1) \geq 0$ weakly, it is clear that $-A(t+1)$ is subparabolic for any constant $A \geq 0$. Thus, since

$$u = [u - A(t+1)] - [-A(t+1)],$$

it is necessary to show that $v \equiv u - A(t+1)$ is subparabolic for A sufficiently large. Since u is a polynomial, this will follow if $L[u - A(t+1)] \leq 0$ weakly on U . Let $w \in C_0^1(U)$ with $w \geq 0$. Then by assumption (iv) on $c + \{d_j\}_{,j}$ it follows that

$$\begin{aligned} 0 &\leq \iint_U [c + \{d_j\}_{,j}] [u - A(t+1)] w \, dx \, dt \\ &= \iint_U \{cw[u - A(t+1)] - d_j w_{,j} [u - A(t+1)] - d_j u_{,j} w\} \, dx \, dt. \end{aligned}$$

Therefore

$$\begin{aligned} &\iint_U \{a_{ij} u_{,i} w_{,j} + d_j w_{,j} [u - A(t+1)] - b_j u_{,j} w \\ &\quad - c[u - A(t+1)] w - [u - A(t+1)] w_t\} \, dx \, dt \\ &\leq \iint_U \{a_{ij} u_{,i} w_{,j} - (b_j + d_j) u_{,j} w + (u_t - A) w\} \, dx \, dt \\ &\leq \iint_U \left\{ \sum M |u_{,i}| |w_{,j}| + |b_j + d_j| |u_{,j}| w + (|u_t| - A) w \right\} \, dx \, dt \\ &= \iint_U \left\{ M \sum (|u_{,i}|)_{,j} + |b_j + d_j| |u_{,j}| + |u_t| - A \right\} w \, dx \, dt \\ &\leq 0 \end{aligned}$$

by choosing A large enough since u is a polynomial and property (v) holds. By choosing $A > \max_U u$ it follows that $u - A(t+1)$ is subparabolic.

The main result of this paper is the following existence and uniqueness theorem:

THEOREM 3. Assume L satisfies properties (i)–(v). If f is a continuous real-valued function on the compact parabolic boundary $\partial_p U$ of the bounded domain U , then f is resolute.

PROOF. Since $\partial_p U$ is compact, let V be an open set containing $\bar{U}$ and apply Lemma 6 to obtain a function u which satisfies

(i) $u = v - w$ where v and w are subparabolic and continuous on V ,

(ii) $\sup_{\partial_p U} |u - f| < \epsilon$ for prescribed $\epsilon > 0$.

Then, by Lemma 5, $v|_{\partial_p U}$ and $w|_{\partial_p U}$ are resolute boundary functions and hence $u|_{\partial_p U} = v|_{\partial_p U} - w|_{\partial_p U}$ is resolute. It follows from Lemma 4 that f is resolute.

REFERENCES

1. Neil Eklund, *Existence and representation of solutions of parabolic equations*, Proc. Amer. Math. Soc. **47** (1975), 137-142.
2. ———, *Generalized super-solutions of parabolic equations*, Trans. Amer. Math. Soc. **220** (1976), 235-242.
3. ———, *Convergent nets of parabolic and generalized super-parabolic functions*, Proc. Amer. Math. Soc. **50** (1975), 237-243.

DEPARTMENT OF MATHEMATICS, CENTRE COLLEGE OF KENTUCKY, DANVILLE, KENTUCKY 40422