ON AN EXTREMAL PROBLEM

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ABSTRACT. Let $X = (x_1, x_2, \dots, x_N)$, $f: \mathbf{R} \rightarrow \mathbf{C}$ and let $\mathbf{P}_n$ be the class of polynomials of degree at most n . The generalized Christoffel function Λ_n corresponding to the measure $d\alpha$ is defined by

$$\Lambda_n(X; f, N, d\alpha) = \min_{\substack{\pi \in \mathbf{P}_{n-1} \\ \pi(x_i) = f(x_i) \\ i=1, 2, \dots, N}} \int_{-\infty}^{\infty} |\pi(t)|^2 d\alpha(t).$$

It is shown that if α satisfies some rather weak conditions then $\lim_{n \rightarrow \infty} n\Lambda_n(X; f, N, d\alpha)$ exists and the limit is also evaluated.

The purpose of the present note is to investigate the asymptotic behavior of the functions $\Lambda_n: \mathbf{R}^N \rightarrow \mathbf{R}$ defined by

$$\Lambda_n(X; f, N, d\alpha) = \min_{\substack{\pi \in \mathbf{P}_{n-1} \\ \pi(x_i) = f(x_i) \\ i=1, 2, \dots, N}} \int_{-\infty}^{\infty} |\pi(t)|^2 d\alpha(t).$$

Here $X = (x_1, x_2, \dots, x_N)$, $f: \mathbf{R} \rightarrow \mathbf{C}$ is a fixed and almost everywhere finite function, $\mathbf{P}_n$ is the set of all polynomials π of a degree at most n and α is a weight function, that is α is nondecreasing on $\mathbf{R}$, it has infinitely many points of increase and every polynomial π belongs to $L_{d\alpha}^2$. Therefore Λ_n is defined and finite for almost every $X \in \mathbf{R}^N$.

Estimates for Λ_n lead to several results in probability theory, statistics and in the theory of orthogonal polynomials. (See [1], [2] and [4].) In fact, it is not hard to explicitly compute Λ_n [3] but the formula for Λ_n is so complicated that it cannot be used to estimate Λ_n when α is not nice. It will be shown that $\lim_{n \rightarrow \infty} n\Lambda_n(X; f, N, d\alpha)$ exists under rather weak assumptions on α and the corresponding limit will also be calculated.

Let $\{p_n(d\alpha)\}_{n=0}^{\infty}$ be the system of polynomials which is orthogonal with respect to $d\alpha$, that is $p_n(d\alpha, x) = \gamma_n x^n + \dots$ with $\gamma_n > 0$ and

$$\int_{-\infty}^{\infty} p_n(d\alpha, t) p_m(d\alpha, t) d\alpha(t) = \delta_{nm}.$$

Let M denote the class of those weights α for which

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} t p_n^2(d\alpha, t) d\alpha(t) = 0$$

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and

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} t p_{n-1}(d\alpha, t) p_n(d\alpha, t) d\alpha(t) = \frac{1}{2}.$$

Let us remark M contains many weights. If, for instance, $\text{supp}(d\alpha) = [-1, 1]$ and $\log \alpha'(\cos \theta) \in L^1$ then $\alpha \in M$ [5]. Furthermore, if α is absolutely continuous,

$$\text{supp}(d\alpha) = [-1, 1] \quad \text{and} \quad \alpha'(t) = \varphi(t) \exp\{-(1-t^2)^{-1/2}\}$$

where $0 < c \leq \varphi(t)$ for $-1 \leq t \leq 1$ and φ is Riemann integrable then also $\alpha \in M$ [4]. Another example is the Pollaczek weight [5]. In the following Δ denotes an interval. It is known that if $\alpha \in M$ then $\Delta \subset \text{supp}(d\alpha)$ iff $\Delta \subset [-1, 1]$ [4].

THEOREM. Let $\alpha \in M$, $\Delta \subset \text{supp}(d\alpha)$, $1/\alpha' \in L^1(\Delta)$. Suppose that there exists a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ with $\varepsilon_k > 0$, $\varepsilon_k \searrow 0$ such that for every fixed k the function Ψ_k , which is defined by

$$\Psi_k(t) = [(1 - \varepsilon_k)^2 - t^2]^{-1/2} \log \alpha'(t),$$

belongs to $L^1(-1 + \varepsilon_k, 1 - \varepsilon_k)$. Then for almost every $X \in \Delta^N$

$$\lim_{n \rightarrow \infty} n \Lambda_n(X; f, N, d\alpha) = \pi \sum_{i=1}^N |f(x_i)|^2 \alpha'(x_i) \sqrt{1 - x_i^2}.$$

For $N = 1$ the Theorem has been proved in [4]. Combining the case $N = 1$ with the following Lemma we immediately obtain the Theorem for every N .

LEMMA. If $\alpha \in M$ then

$$\lim_{n \rightarrow \infty} \frac{\Lambda_n(X; f, N, d\alpha)}{\sum_{i=1}^N \Lambda_n(x_i; |f(x_i)|, 1, d\alpha)} = 1$$

for almost every $X \in [\text{supp}(d\alpha)]^N$.

PROOF. If $\pi \in \mathbf{P}_{n-1}$ then π can be expressed as

$$\pi(x) = \int_{-\infty}^{\infty} \pi(t) K_n(d\alpha, x, t) d\alpha(t)$$

where

$$K_n(d\alpha, x, t) = \sum_{k=0}^{n-1} p_k(d\alpha, x) p_k(d\alpha, t).$$

If X is given and $\prod_{i < j} (x_i - x_j) \neq 0$ then we can write

$$\begin{aligned} & \sum_{i=1}^N \Lambda_n(x_i; |\pi(x_i)|, 1, d\alpha) \\ &= \int_{-\infty}^{\infty} \pi(t) \left[\sum_{i=1}^N \text{sign}\{\pi(x_i)^2\} \pi(x_i) \Lambda_n(x_i; 1, 1, d\alpha) K_n(d\alpha, x_i, t) \right] d\alpha(t). \end{aligned}$$

Using Cauchy's inequality, orthogonality relations and the well-known fact that

$$K_n(d\alpha, x, x) = \Lambda_n(x; 1, 1, d\alpha)^{-1}, \quad (1)$$

we obtain

$$\begin{aligned} & \left[\sum_{i=1}^N \Lambda_n(x_i; |\pi(x_i)|, 1, d\alpha) \right]^2 \\ & \leq \int_{-\infty}^{\infty} |\pi(t)|^2 d\alpha(t) \cdot \left[\sum_{i=1}^N \Lambda_n(x_i; |\pi(x_i)|, 1, d\alpha) + 2 \sum_{j < i} |\pi(x_i)| |\pi(x_j)| \right. \\ & \quad \left. \cdot \Lambda_n(x_i; 1, 1, d\alpha) \Lambda_n(x_j; 1, 1, d\alpha) |K_n(d\alpha, x_i, x_j)| \right]. \end{aligned}$$

If $\alpha \in M$ then clearly $\text{supp}(d\alpha)$ is compact. Therefore by the Christoffel-Darboux formula there exists a number C depending on X and $\text{supp}(d\alpha)$ such that

$$|K_n(d\alpha, x_i, x_j)| \leq C [|p_n(d\alpha, x_i) p_{n-1}(d\alpha, x_j)| + |p_{n-1}(d\alpha, x_i) p_n(d\alpha, x_j)|].$$

Hence

$$\begin{aligned} & \left[\sum_{i=1}^N \Lambda_n(x_i; |\pi(x_i)|, 1, d\alpha) \right]^2 \\ & \leq \int_{-\infty}^{\infty} |\pi(t)|^2 d\alpha(t) \left[\sum_{i=1}^N \Lambda_n(x_i; |\pi(x_i)|, 1, d\alpha) \right. \\ & \quad + 4C \sum_{i=1}^N |\pi(x_i)| |p_n(d\alpha, x_i)| \Lambda_n(x_i; 1, 1, d\alpha) \\ & \quad \left. \cdot \sum_{i=1}^N |\pi(x_i)| |p_{n-1}(d\alpha, x_i)| \Lambda_n(x_i; 1, 1, d\alpha) \right]. \end{aligned}$$

Formula (1) implies that $p_{n-1}^2(d\alpha, x_i) \Lambda_n(x_i; 1, 1, d\alpha) \leq 1$. Consequently by Cauchy's inequality

$$\begin{aligned} & \left[\sum_{i=1}^N \Lambda_n(x_i; |\pi(x_i)|, 1, d\alpha) \right]^2 \\ & \leq \int_{-\infty}^{\infty} |\pi(t)|^2 d\alpha(t) \sum_{i=1}^N \Lambda_n(x_i; |\pi(x_i)|, 1, d\alpha) \\ & \quad \cdot \left\{ 1 + 4C\sqrt{N} \left[\sum_{i=1}^N \Lambda_n(x_i; |p_n(d\alpha, x_i)|, 1, d\alpha) \right]^{1/2} \right\}, \end{aligned}$$

that is

$$\sum_{i=1}^N \Lambda_n(x_i; |\pi(x_i)|, 1, d\alpha) \leq \int_{-\infty}^{\infty} |\pi(t)|^2 d\alpha(t) \cdot \left\{ 1 + 4C\sqrt{N} \left[\sum_{i=1}^N \Lambda_n(x_i; |p_n(d\alpha, x_i)|, 1, d\alpha) \right]^{1/2} \right\}.$$

This inequality holds for each $X \in \mathbf{R}^N$ such that $\prod_{i < j} (x_i - x_j) \neq 0$. Taking the minimum on the right side over every polynomial $\pi \in \mathbf{P}_{n-1}$ such that $\pi(x_i) = f(x_i)$ ($i = 1, 2, \dots, N$) we obtain

$$\sum_{i=1}^N \Lambda_n(x_i; |f(x_i)|, 1, d\alpha) \leq \Lambda_n(X; f, N, d\alpha) \cdot \left\{ 1 + 4C\sqrt{N} \left[\sum_{i=1}^N \Lambda_n(x_i; |p_n(d\alpha, x_i)|, 1, d\alpha) \right]^{1/2} \right\}.$$

It has been proved in [4] that if $\alpha \in M$ then for every $x \in \text{supp}(d\alpha)$

$$\lim_{n \rightarrow \infty} \Lambda_n(x; |p_n(d\alpha, x)|, 1, d\alpha) = 0. \quad (2)$$

Consequently, if $\alpha \in M$ then for almost every $X \in [\text{supp}(d\alpha)]^N$

$$\sum_{i=1}^N \Lambda_n(x_i; |f(x_i)|, 1, d\alpha) \leq [1 + \sigma(1)] \Lambda_n(X; f, N, d\alpha) \quad (3)$$

where $\lim_{n \rightarrow \infty} \sigma(1) = 0$ for almost every $X \in [\text{supp}(d\alpha)]^N$. Our next aim is to establish the converse inequality. For a given X let π^X be defined by

$$\pi^X(t) = \sum_{i=1}^N f(x_i) \Lambda_{n-N+1}(x_i; 1, 1, d\alpha) K_{n-N+1}(d\alpha, t, x_i) l_i(t)$$

where $\{l_i\}_{i=1}^N$ are the fundamental polynomials of Lagrange interpolation corresponding to $\{x_i\}_{i=1}^N$. Clearly $\pi^X \in \mathbf{P}_{n-1}$ for almost every $X \in \mathbf{R}^N$. Furthermore, if $\pi^X \in \mathbf{P}_{n-1}$ then $\pi^X(x_i) = f(x_i)$ for $i = 1, 2, \dots, N$. Let us compute the $L_{d\alpha}^2$ norm of π^X . Introducing the notation

$$G_n(d\alpha, g, x) = \Lambda_n(x; 1, 1, d\alpha) \int_{-\infty}^{\infty} g(t) K_n^2(d\alpha, x, t) d\alpha(t)$$

we have

$$\begin{aligned} \int_{-\infty}^{\infty} |\pi^X(t)|^2 d\alpha(t) &= \sum_{i=1}^N |f(x_i)|^2 \Lambda_{n-N+1}(x_i; 1, 1, d\alpha) G_{n-N+1}(d\alpha, l_i^2, x_i) \\ &+ 2 \sum_{i < j} \text{Re} [f(x_i) \overline{f(x_j)}] \Lambda_{n-N+1}(x_i; 1, 1, d\alpha) \Lambda_{n-N+1}(x_j; 1, 1, d\alpha) \\ &\cdot \int_{-\infty}^{\infty} l_i(t) l_j(t) K_{n-N+1}(d\alpha, t, x_i) K_{n-N+1}(d\alpha, t, x_j) d\alpha(t). \end{aligned} \quad (4)$$

If $\alpha \in M$, g is continuous on $\text{supp}(d\alpha)$ and $x \in \text{supp}(d\alpha)$ then

$$\lim_{n \rightarrow \infty} G_n(d\alpha, g, x) = g(x)$$

(see [4]). In particular, if X is such that $\prod_{i < j} (x_i - x_j) \neq 0$ then

$$\lim_{n \rightarrow \infty} G_n(d\alpha, l_i^2, x_i) = l_i^2(x_i) = 1.$$

Let

$$I_{ij} = \int_{-\infty}^{\infty} l_i(t) l_j(t) K_{n-N+1}(d\alpha, t, x_i) K_{n-N+1}(d\alpha, t, x_j) d\alpha(t).$$

If $\prod_{k < l} (x_k - x_l) \neq 0$ then $l_i l_j \in \mathbf{P}_{2N-2}$. By a direct calculation we obtain that

$$\begin{aligned} I_{ij} = & \int_{-\infty}^{\infty} l_i(t) l_j(t) \left[K_{n-3N+3}(d\alpha, x_i, t) + \sum_{l=n-3N+3}^{n-N} p_k(d\alpha, x_i) p_k(d\alpha, t) \right] \\ & \cdot K_{n-N+1}(d\alpha, x_j, t) d\alpha(t) = l_i(x_j) l_j(x_j) K_{n-3N+3}(d\alpha, x_i, x_j) \\ & + \sum_{k=n-3N+3}^{n-N} p_k(d\alpha, x_i) \sum_{l=k-2N+2}^{n-N} p_l(d\alpha, x_j) \\ & \cdot \int_{-\infty}^{\infty} l_i(t) l_j(t) p_k(d\alpha, t) p_l(d\alpha, t) d\alpha(t). \end{aligned}$$

Since $l_i(x_j) = \delta_{ij}$, we get $i < j$

$$|I_{ij}| \leq C_1 \sum_{k=n-3N+3}^{n-N} |p_k(d\alpha, x_i)| \cdot \sum_{l=k-2N+2}^{n-N} |p_l(d\alpha, x_j)| \quad (5)$$

where

$$C_1 = \max_{\substack{i=1,2,\dots,N \\ t \in \text{supp}(d\alpha)}} |l_i(t)|^2.$$

Using the recurrence formula which the orthogonal polynomials satisfy it can easily be seen that for $\alpha \in M$ (5) implies

$$|I_{ij}| \leq C_2 \sum_{k=n-N-1}^{n-N} |p_k(d\alpha, x_i)| \sum_{l=n-N-1}^{n-N} |p_l(d\alpha, x_j)|$$

where C_2 is independent of n . Now applying (1) and (2) the second sum on the right-hand side of (4) can easily be estimated. We get

$$\int_{-\infty}^{\infty} |\pi^X(t)|^2 d\alpha(t) = [1 + \sigma(1)] \sum_{i=1}^N |f(x_i)|^2 \Lambda_{n-N+1}(x_i; 1, 1, d\alpha)$$

for almost every $X \in [\text{supp}(d\alpha)]^N$. It has been proved in [4] that if $\alpha \in M$ and $x \in \text{supp}(d\alpha)$ then

$$\lim_{n \rightarrow \infty} \Lambda_{n-N+1}(x; 1, 1, d\alpha) \Lambda_n(x; 1, 1, d\alpha)^{-1} = 1$$

for every fixed N . Thus

$$\Lambda_n(X; f, N, d\alpha) \leq [1 + \sigma(1)] \sum_{i=1}^N \Lambda_n(x_i; |f(x_i)|, 1, d\alpha) \quad (6)$$

for almost every $X \in [\text{supp}(d\alpha)]^N$. The Lemma follows now from estimates (3) and (6).

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