CONVERGENCE OF MEASURABLE RANDOM FUNCTIONS

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ABSTRACT. Using the theorem of Fréchet and Kolmogorov about compact subsets of the space $L_p[0, 1]$ and Prohorov's theorem about the convergence of measures defined on a complete metric space we proved in [2] the following theorem: Let $\xi_1(t), \dots, \xi_n(t), \dots$ and $\xi(t)$ be measurable random processes ($0 < t < 1$) and suppose that there exist numbers C and $p > 1$ such that $E|\xi_n(t)|^p < C$ for all n and t . If $E|\xi_n(t)|^p \rightarrow E|\xi(t)|^p$ for all t and if for any finite set $\{t_1, \dots, t_k\} \subset [0, 1]$ the joint distribution of $\xi_n(t_1), \dots, \xi_n(t_k)$ converges to the joint distribution of $\xi(t_1), \dots, \xi(t_k)$, then the distribution of $f(\xi_n)$ converges to the distribution of $f(\xi)$ for any continuous functional f on $L_p[0, 1]$. In this paper this theorem is generalized to random measurable functions. The results of the present paper are related to the results of [1], [3], [4].

Let (X, ρ) be a metric space. We call the set

$$U_y^r = \{x \in X: \rho(y, x) < r\} \quad (r > 0)$$

a ball. We say that a set U is a domain if U is the union of a countable set of balls. We shall consider on X only probability measures for which any ball is a measurable set. By $\rho_y(x)$ we denote the function $x \mapsto \rho(y, x)$.

LEMMA. Let $\mu_1, \dots, \mu_n, \dots$, and μ be measures defined on X such that for any finite set $\{y_1, \dots, y_k\} \subset X$ the joint distribution of $\rho_{y_1}(x), \dots, \rho_{y_k}(x)$ with respect to the measure μ_n converges to the joint distribution of $\rho_{y_1}(x), \dots, \rho_{y_k}(x)$ with respect to the measure μ . If U is a domain and there exists a domain V such that $U \cap V = \emptyset$ and $\mu(U) + \mu(V) = 1$, then $\mu_n(U) \rightarrow \mu(U)$.

PROOF. By definition $U = \bigcup_{i=1}^{\infty} U_{y_i}^{r_i}$, $V = \bigcup_{i=1}^{\infty} U_{z_i}^{s_i}$. Let $\varepsilon > 0$. We will find numbers $k, d > 0$ such that

$$\mu(U^* \cup V^*) > 1 - \varepsilon, \quad \text{where } U^* = \bigcup_{i=1}^k U_{y_i}^{r_i-d}, \quad V^* = \bigcup_{i=1}^k U_{z_i}^{s_i-d};$$

$$\mu(U^*) = \mu(\bar{U}^*), \quad \mu(V^*) = \mu(\bar{V}^*).$$

From the conditions of the lemma we obtain:

$$\mu_n(U^*) \rightarrow \mu(U^*), \quad \mu_n(V^*) \rightarrow \mu(V^*).$$

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But $U^* \subset U$, $V^* \subset V$, $\mu(U) - \mu(U^*) < \varepsilon$, $\mu(V) - \mu(V^*) < \varepsilon$, $\mu(U) + \mu(V) = 1$. Since ε is arbitrary we have $\mu_n(U) \rightarrow \mu(U)$. $\square$

Let $T = \{t\}$ be a space with a probability measure ν , and $\Omega = \{\omega\}$ be a probability space with the probability P . If for any point $t \in T$ a random variable $\xi(t, \omega)$ is defined, then we say that $\xi(t, \omega)$ is a random function; if $\xi(t, \omega)$ is a measurable function on $T \times \Omega$ with respect to the measure $\nu \times P$, then we say that $\xi(t, \omega)$ is a measurable random function.

THEOREM. Let $\xi_1(t), \dots, \xi_n(t), \dots$ and $\xi(t)$ be measurable random functions defined on (T, ν) , and suppose that there exists a function $f \in L_p(T, \nu)$, $p \geq 1$, such that $E|\xi_n(t)|^p \leq f(t)$ for any t and n . Suppose further that $E|\xi_n(t)|^p \rightarrow E|\xi(t)|^p$ for any t and that for any finite set $\{t_1, \dots, t_k\} \subset T$ the joint distribution of $\xi_n(t_1), \dots, \xi_n(t_k)$ converges to the joint distribution of $\xi(t_1), \dots, \xi(t_k)$. Finally, let $\mu_1, \dots, \mu_n, \dots$ and μ be the measures on $L_p(T, \nu)$ defined by our random functions. If U is a domain in $L_p(T, \nu)$ and there exists a domain $V \subset L_p(T, \nu)$ such that $U \cap V = \emptyset$ and $\mu(U) + \mu(V) = 1$, then $\mu_n(U) \rightarrow \mu(U)$.

Before proving this theorem we point out an important proposition which it implies. Suppose that there exists a separable space $Z \subset L_p(T, \nu)$ such that $\mu(Z) = 1$; under this assumption, any open subset of $L_p(T, \nu)$ is measurable with respect to μ . Let U be an open subset of $L_p(T, \nu)$ such that $\mu(U) = \mu(\bar{U})$. Then $\bar{\mu}_n(U) \rightarrow \mu(U)$, $\underline{\mu}_n(U) \rightarrow \mu(U)$, where $\bar{\mu}_n(U)$ and $\underline{\mu}_n(U)$ denote the outer and inner measures of U .

PROOF OF THE THEOREM. Let $\{y_1(t), \dots, y_k(t)\} \subset L_p(T, \nu)$. In view of the lemma it is enough to prove that the joint distribution of

$$\int_T |y_1(t) - \xi_n(t)|^p d\nu, \dots, \int_T |y_k(t) - \xi_n(t)|^p d\nu \quad (1)$$

converges to the joint distribution of

$$\int_T |y_1(t) - \xi(t)|^p d\nu, \dots, \int_T |y_k(t) - \xi(t)|^p d\nu. \quad (2)$$

Let $\chi(\omega)$ be a random variable. For any $s > 0$ we set

$$\chi^{(s)}(\omega) = \begin{cases} 0 & \text{if } |\chi(\omega)| < s, \\ |\chi(\omega)|^p & \text{if } |\chi(\omega)| \geq s. \end{cases}$$

We fix $\varepsilon > 0$. There exist a natural number N , a number $r > 0$ and a measurable set $Q \subset T$ such that

- (a) $E\eta_{ni}^{(r)}(t) < \varepsilon$ if $n \geq N$, $t \in Q$ and where $\eta_{ni} = y_i(t) - \xi_n(t)$;
- (b) $\int_{T \setminus Q} |y_i(t)|^p d\nu < \varepsilon$, $1 \leq i \leq k$;
- (c) $\int_{T \setminus Q} |f(t)|^p d\nu < \varepsilon$.

For any $s > 0$ consider the real function $\varphi_s(x) = \min(|x|^p, s^p)$. It is obvious

that for $n \geq N$,

$$\begin{aligned}
 & E \int_T |y_i(t) - \xi_n(t)|^p d\nu - E \int_T \varphi_r(y_i(t) - \xi_n(t)) d\nu \\
 & \leq E \int_{T \setminus Q} |y_i(t) - \xi_n(t)|^p d\nu + \int_Q E \eta_{n,i}^{(r)}(t) d\nu \\
 & \leq 2^p \int_{T \setminus Q} |y_i(t)|^p d\nu + 2^p \int_{T \setminus Q} |\xi_n(t)|^p d\nu + \varepsilon \\
 & \leq 2^p \cdot \varepsilon + 2^p \int_{T \setminus Q} |f(t)|^p d\nu + \varepsilon \leq \varepsilon(2^{p+1} + 1).
 \end{aligned}$$

If ε is small enough, then for $n \geq N$ the joint distribution of (1) differs little from the joint distribution of

$$\int_T \varphi_r(y_1(t) - \xi_n(t)) d\nu, \dots, \int_T \varphi_r(y_k(t) - \xi_n(t)) d\nu, \quad (3)$$

and the joint distribution of (2) differs little from the joint distribution of

$$\int_T \varphi_r(y_1(t) - \xi(t)) d\nu, \dots, \int_T \varphi_r(y_k(t) - \xi(t)) d\nu. \quad (4)$$

Therefore it is enough to prove that the joint distribution of (3) converges to the joint distribution of (4). Let $m_1, \dots, m_k$ be any set of k natural numbers. We have

$$\begin{aligned}
 & E \prod_{i=1}^k \left(\int_T \varphi_r(y_i(t) - \xi_n(t)) d\nu \right)^{m_i} \\
 & = \int_{T^*} E \prod_{i=1}^k \prod_{j=1}^{m_i} \varphi_r(y_i(t_{ij}) - \xi_n(t_{ij})) d\nu^* \\
 & \rightarrow \int_{T^*} E \prod_{i=1}^k \prod_{j=1}^{m_i} \varphi_r(y_i(t_{ij}) - \xi(t_{ij})) d\nu^* \\
 & = E \prod_{i=1}^k \left(\int_T \varphi_r(y_i(t) - \xi(t)) d\nu \right)^{m_i},
 \end{aligned}$$

where T^* is the product of $\sum_{i=1}^k m_i$ copies of the space T and the measure ν^* is the product of $\sum_{i=1}^k m_i$ measures ν . From this it follows immediately that the joint distribution of (3) converges to the joint distribution of (4). $\square$

In classical measure theory the basic theorem is the following:

THEOREM OF LEBESGUE. *Let $\{\varphi_n(t)\} \subset L_1(T, \nu)$, $\varphi_n(t) \rightarrow \varphi(t)$ for all t , and suppose there exists $f \in L_1(T, \nu)$ such that $|\varphi_n(t)| \leq f(t)$ for all t . Then $\varphi(t) \in L_1(T, \nu)$ and $\|\varphi_n - \varphi\| \rightarrow 0$.*

It is possible to say that the theorem proved in this paper is a reasonable infinite generalization of the theorem of Lebesgue.

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