A NOTE ON S -CLOSED SPACES

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ABSTRACT. A topological space X is said to be S -closed if and only if for every semi-open cover of X there exists a finite subfamily such that the union of their closures cover X . For a Hausdorff space, the concept of S -closed is shown to be equivalent to the concept of extremally disconnected and nearly compact. Further it has been shown that EDH-closed spaces are precisely S -closed Hausdorff spaces.

T. Thompson introduced and studied S -closed spaces in [6]. A space X is said to be S -closed if every semi-open cover of X admits a finite subfamily, the closures of whose members cover the space, where a set S is semi-open if and only if there exists an open set V such that $V \subset A \subset \bar{V}$.

Since every regularly closed set is semi-open and the closure of semi-open sets is regularly closed the following result is immediate.

THEOREM 1. *The following are equivalent for a topological space X :*

- (a) X is S -closed.
- (b) Every regularly closed cover has a finite subcover.
- (c) Every family of regularly open sets having the finite intersection property has nonempty intersection.

Since every S -closed Hausdorff space is extremally disconnected [6, Theorem 7] and in extremally disconnected spaces regularly closed sets coincide with clopen sets, the following corollary follows from the above theorem.

COROLLARY 2. *The following are equivalent for a Hausdorff space X :*

- (a) X is S -closed.
- (b) Every regularly closed cover has a finite disjoint clopen refinement.

The main result obtained by T. Thompson is that for compact Hausdorff or compact regular spaces, the concept of S -closed is equivalent to the concept of extremally disconnected. Since in a regular space every open cover can be refined by a regularly closed cover it is immediate from Theorem 1(b)

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that a regular S -closed space is compact. Thus for regular spaces, Thompson's result gives that the concept of S -closed is equivalent to the concept of extremally disconnected and compact. However, the situation is not the same in case of Hausdorff spaces inasmuch as S -closed Hausdorff spaces are not necessarily compact. But before we could give an example of such a space we need some more results about S -closed spaces. First a couple of definitions.

A space $(X, \mathcal{T})$ is nearly compact [5] if every regularly open cover has a finite subcover and is almost regular [4] if for each point x and a regularly open set G there exists an open set H such that $x \in H \subset \overline{H} \subset G$.

It is immediate from these definitions that in an almost regular space every regularly open cover can be refined by regularly closed sets and hence every almost regular S -closed space is nearly compact in view of Theorem 1(b). Using essentially the same proof as that of Theorems 5 and 6 of [6] one can show that every extremally disconnected nearly compact space is S -closed and every almost regular S -closed space is extremally disconnected. Thus we have:

THEOREM 3. *An almost regular space is S -closed if and only if it is nearly compact and extremally disconnected.*

It may be noted that since almost regular S -closed spaces are extremally disconnected, Corollary 2 remains true even for almost regular spaces. Also note that since regular $\Leftrightarrow$ almost regular + semiregular [4] and compact $\Leftrightarrow$ nearly compact + semiregular [5], Thompson's result follows from Theorem 3.

In view of Theorem 7 of [6] and the fact that extremally disconnected spaces are almost regular, the following is immediate from Theorem 3.

THEOREM 4. *An Hausdorff space is S -closed if and only if it is nearly compact and extremally disconnected.*

Robert A. Herrmann [2] constructed a nearly compact Hausdorff extension αX for almost completely regular spaces, where a space X is almost completely regular if for each point $x \in X$ and a regularly closed set $A \ni x$ there exists a continuous map $f: X \rightarrow [0, 1]$ such that $f(x) = 0$ and $f(A) = \{1\}$. It was observed in [2] that αX is compact if and only if X is compact. Further in [1] it has been proved that αX is extremally disconnected if and only if X is extremally disconnected.

Now if $\mathbb{N}$ denotes the set of natural numbers with discrete topology, then $\alpha\mathbb{N}$ is extremally disconnected and hence an S -closed space in view of Theorem 4. However, since $\mathbb{N}$ is not compact, $\alpha\mathbb{N}$ is not compact. Thus there exist S -closed Hausdorff spaces which are not compact.

However since nearly compact + semiregular $\Rightarrow$ compact [5] we have immediately the following corollary to Theorem 4.

COROLLARY 5. *The following are equivalent for a Hausdorff space X :*

(a) *X is S -closed and semiregular.*

- (b) X is compact and extremally disconnected.
- (c) X is S -closed and regular.

An extremally disconnected Hausdorff space is an EDH-closed space if it is closed in any extremally disconnected Hausdorff space in which it can be embedded. S -closed Hausdorff spaces happen to characterize these spaces as shown by the following theorem.

THEOREM 6. *The following are equivalent for a space X :*

- (a) X is EDH-closed.
- (b) X is S -closed and Hausdorff.
- (c) X is nearly compact, Hausdorff and extremally disconnected.
- (d) X is H -closed and extremally disconnected.

PROOF. (a) $\Rightarrow$ (b). Since the semiregularization space of an extremally disconnected space is completely regular, every extremally disconnected space is almost completely regular. Thus any given extremally disconnected Hausdorff space X can be densely embedded in a nearly compact Hausdorff space αH , which is extremally disconnected since X is extremally disconnected and hence is S -closed in view of Theorem 4. Now if X is EDH-closed, then X is a closed subspace of X . Since it is already dense in X , this implies $X = \alpha X$ and hence X is S -closed.

(b) $\Rightarrow$ (c) follows from Theorem 4 and (c) $\Rightarrow$ (d) $\Rightarrow$ (a) are obvious.

It is worth noting that EDH-closed spaces are not necessarily compact or semiregular as αN is an EDH-closed space in view of Theorem 6 but is neither compact nor semiregular. Further it is pointed out that in [3] it was mentioned that (a) and (d) are equivalent and using that equivalence an alternate proof could be provided to Theorem 6.

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