TWISTED WHITEHEAD PRODUCTS

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ABSTRACT. By combining the Hopf construction and the Whitehead product map construction we introduce twisted Whitehead products and then extend several well-known properties of Whitehead products to the twisted situation.

1. Introduction. Universally defined homotopy pairings $\pi_1^A(X) \times \pi_1^B(X) \rightarrow \pi_1^Y(X)$ are classified by the elements of the group $\pi_1^Y(\Sigma A \vee \Sigma B)$. Those elements in the image of the boundary homomorphism

$$\pi_2^Y(\Sigma A \times \Sigma B, \Sigma A \vee \Sigma B) \rightarrow \pi_1^Y(\Sigma A \vee \Sigma B)$$

correspond to the $\gamma^\dagger$ -homotopy operations of Hilton [4, p. 189] and have the property that they vanish under suspension. For $Y = A \wedge B$, the generalized Whitehead product $[\alpha, \beta] \in \pi_1^{A \wedge B}(X)$ of $\alpha \in \pi_1^A(X)$ and $\beta \in \pi_1^B(X)$ is an example of such an operation.

Alternatively the $\gamma^\dagger$ -homotopy operations can be characterized as those classes $\gamma: \Sigma Y \rightarrow \Sigma A \vee \Sigma B$ which are trivial on both ΣA and ΣB . Recent work of H. J. Baues [3, p. 171] focuses attention as well on classes $\gamma: \Sigma Y \rightarrow \Sigma A \vee \Sigma B$ which are trivial on ΣA but not necessarily on ΣB . For $Y = A \wedge B$ we shall provide here examples of these latter operations by introducing twisted Whitehead products. Our purpose is to generalize some well-known properties of the usual Whitehead product to the twisted case. For properties of the usual Whitehead product the reader is referred to the classic article of Arkowitz [1].

2. Notation. With one exception we work entirely within the based topological category. Furthermore we assume that all spaces are well pointed; i.e., the inclusion of the base point is an (unbased) cofibration. All homotopies preserve base points.

$A * B$ denotes the unreduced topological join of A and B (this is the one exception above) and is provided with the variable base point as explained in [5]. We recall that since all spaces are well pointed, the composite

$$\nu: A * B \xrightarrow{\kappa} \Sigma(A \times B) \xrightarrow{\Sigma q} \Sigma(A \wedge B)$$

is a (based) homotopy equivalence. (Here κ collapses the “ends” of the join

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and $q: A \times B \rightarrow A \wedge B$ is the usual quotient.) We use ν to regard $A * B$ as an H -cogroup.

Given a map $f: A \times B \rightarrow C$ the *Hopf construction on f* is the map $h(f): A * B \rightarrow \Sigma C$ given by

$$h(f)[a, b, t] = [f(a, b), t], \quad a \in A, b \in B, 0 \leq t \leq 1.$$

By means of ν , $h(f)$ may be regarded as a class $h(f): \Sigma(A \wedge B) \rightarrow \Sigma C$. Suppose that f has type (e, d) . That is, $f|_{A \times \{*\}} \simeq e: A \rightarrow C$ and $f|_{\{*\} \times B} \simeq d: B \rightarrow C$. Then one easily verifies:

(2.1) Regarding $h(f) \in \pi_1^{A \wedge B}(\Sigma C)$ we have

$$h(f)\Sigma q \simeq -\Sigma e \Sigma \text{pr}_A + (\Sigma f - \Sigma d \Sigma \text{pr}_B)$$

where pr_A and pr_B are the two projection maps on $A \times B$.

(2.2) We shall also need the action of $\pi_1^A(Z)$ on $\pi_1^{A \wedge B}(Z)$ which is defined as follows. Given $\alpha \in \pi_1^A(Z)$ and $\xi \in \pi_1^{A \wedge B}(Z)$, $\xi^\alpha \in \pi_1^{A \wedge B}(Z)$ is defined by

$$\xi^\alpha \Sigma q \simeq -\alpha \Sigma \text{pr}_A + \xi \Sigma q + \alpha \Sigma \text{pr}_B.$$

3. The twisted Whitehead product map. From [1] we recall that given spaces A and B the (generalized) *Whitehead product map* $W: A * B \rightarrow \Sigma A \vee \Sigma B$ is defined by the formula

$$W[a, b, t] = \begin{cases} (*, [b, 1 - 2t]), & 0 \leq t \leq 1/2, \\ ([a, 2t - 1], *), & 1/2 \leq t \leq 1. \end{cases}$$

By combining the Whitehead product map construction and the Hopf construction we obtain the twisted Whitehead product map.

(3.1) **DEFINITION.** For a fixed map $f: A \times B \rightarrow B$ the *twisted Whitehead product map* $W_f: A * B \rightarrow \Sigma A \vee \Sigma B$ is defined by

$$W_f[a, b, t] = \begin{cases} (*, [f(a, b), 2t]), & 0 \leq t \leq 1/2, \\ ([a, 2 - 2t], *), & 1/2 \leq t \leq 1. \end{cases}$$

Observe that W_f is the composite

$$\begin{aligned} A * B &\xrightarrow{\kappa} \Sigma(A \times B) \xrightarrow{r} \Sigma(A \times B) \\ &\xrightarrow{\omega} \Sigma(A \times B) \vee \Sigma(A \times B) \xrightarrow{\Sigma \text{pr}_A \vee -\Sigma f} \Sigma A \vee \Sigma B \end{aligned}$$

where r denotes reversal of parameter and ω is the suspension comultiplication. Thus the class of W_f depends only on the class of f .

(3.2) (i) If $f = \text{pr}_B: A \times B \rightarrow B$ then the composite

$$A * B \xrightarrow{W_f} \Sigma A \vee \Sigma B \xrightarrow{r \vee r} \Sigma A \vee \Sigma B$$

is the generalized Whitehead product map for A and B .

(ii) On ΣA , W_f is trivial and on ΣB , W_f is the Hopf construction on f . That

is, the diagram

$$\begin{array}{ccc} & & \Sigma A \times \Sigma B \\ & \nearrow 0 \Delta h(f) & \\ A * B & \xrightarrow{W_f} & \Sigma A \vee \Sigma B \end{array}$$

is homotopy commutative.

Since $\Sigma(\Sigma A \times \Sigma B)$ homotopy retracts onto $\Sigma(\Sigma A \vee \Sigma B)$, (3.2) (ii) implies:

(3.3) PROPOSITION. *The diagram*

$$\begin{array}{ccc} \Sigma(A * B) & \xrightarrow{\Sigma W_f} & \Sigma(\Sigma A \vee \Sigma B) \\ & \searrow \Sigma h(f) & \nearrow \Sigma \text{inc}_{\Sigma B} \\ & \Sigma^2 B & \end{array}$$

is homotopy commutative.

Observe that since $h(\text{pr}_B) \simeq 0$, (3.3) reduces to $\Sigma W \simeq 0$ in the untwisted case.

(3.4) DEFINITION. For any space Z the twisted Whitehead product pairing

$$[\ , \]_f: \pi_1^A(Z) \times \pi_1^B(Z) \rightarrow \pi(A * B, Z)$$

is defined by $[\alpha, \beta]_f = W_f(\alpha \nabla \beta)$ for $\alpha \in \pi_1^A(Z)$, $\beta \in \pi_1^B(Z)$.

By (3.2)(i) and (3.3), we have

$$\begin{aligned} [\alpha, \beta]_{\text{pr}_B} &= [-\alpha, -\beta] \text{ (usual Whitehead product),} \\ \Sigma[\alpha, \beta]_f &= \Sigma(\beta h(f)). \end{aligned}$$

4. The commutator definition. The untwisted Whitehead product pairing has an alternative definition in terms of commutators [1, Theorem 2.4]. The twisted analogue is given in the following theorem.

(4.1) THEOREM. Suppose $f: A \times B \rightarrow B$ has type (e, d) ; $\alpha \in \pi_1^A(Z)$, $\beta \in \pi_1^B(Z)$. Then

$$[\alpha, \beta]_f \equiv (\beta \Sigma f - \alpha \Sigma \text{pr}_A) \kappa \simeq \theta \kappa$$

where

$$\theta \equiv (\alpha \Sigma \text{pr}_A - \beta \Sigma e \Sigma \text{pr}_A) + ((\beta \Sigma f - \alpha \Sigma \text{pr}_A) - \beta \Sigma d \Sigma \text{pr}_B).$$

Hence regarding $[\alpha, \beta]_f \in \pi_1^{A \wedge B}(Z)$ we have

$$[\alpha, \beta]_f \Sigma q \simeq (\alpha \Sigma \text{pr}_A - \beta \Sigma e \Sigma \text{pr}_A) + ((\beta \Sigma f - \alpha \Sigma \text{pr}_A) - \beta \Sigma d \Sigma \text{pr}_B).$$

(4.2) LEMMA. *The co-operator map*

$$c: \Sigma(A \times B) \rightarrow \Sigma A \vee \Sigma B \vee \Sigma(A \times B)$$

associated to the cofibration $A \vee B \rightarrow A * B \xrightarrow{\kappa} \Sigma(A \times B)$ is the composite

$$\begin{aligned} X &\xrightarrow{\omega} X \vee X \xrightarrow{1 \vee \omega} X \vee X \vee X \xrightarrow{1 \vee T} X \vee X \vee X \\ &\xrightarrow{-\Sigma \text{pr}_A \vee \Sigma \text{pr}_B \vee 1} \Sigma A \vee \Sigma B \vee X \end{aligned}$$

where $X = \Sigma(A \times B)$, $\omega =$ suspension comultiplication and T is interchange of factors.

PROOF. A routine check. $\square$

PROOF OF (4.1). Let ψ denote the composite

$$\Sigma A \vee \Sigma B \xrightarrow{\omega \vee 1} \Sigma A \vee \Sigma A \vee \Sigma B \xrightarrow{-\alpha \nabla \beta \Sigma \epsilon \nabla (-\beta \Sigma d)} Z.$$

Then a short computation shows that the composite

$$\Sigma(A \times B) \xrightarrow{c} \Sigma A \vee \Sigma B \vee \Sigma(A \times B) \xrightarrow{\psi \nabla (\beta \Sigma f - \alpha \Sigma \text{pr}_A)} Z$$

with c as in (4.2) is just θ . But this implies $\theta \kappa \simeq (\beta \Sigma f - \alpha \Sigma \text{pr}_A) \kappa = [\alpha, \beta]_f$, as desired. $\square$

(4.3) Since $\text{pr}_B: A \times B \rightarrow B$ has type $(0,1)$, (4.1) and (3.2)(i) show that the untwisted Whitehead product satisfies

$$[\alpha, \beta] \Sigma q \simeq -\alpha \Sigma \text{pr}_A - \beta \Sigma \text{pr}_B + \alpha \Sigma \text{pr}_A + \beta \Sigma \text{pr}_B.$$

$$(4.4) \text{ COROLLARY. } [\alpha, \beta]_f = [\beta h(f)]^{-\alpha} + [-\alpha, -\beta \Sigma d].$$

Note that this corollary expresses the twisted Whitehead product in terms of the untwisted Whitehead product and the action (2.2).

As in [2] let $\pi_1^B(Z) \star \pi_1^{A \wedge B}(Z)$ be the semidirect product with operation

$$(\beta, \xi) \star (\bar{\beta}, \bar{\xi}) = (\beta + \bar{\beta}, \xi^{\bar{\beta}} + \bar{\xi}).$$

Then $\pi_1^A(Z)$ acts on $\pi_1^B(Z) \star \pi_1^{A \wedge B}(Z)$ by $(\beta, \xi)^\alpha = (\beta, -[\alpha, \beta] + \xi^\alpha)$. From (4.4) we have:

(4.5) PROPOSITION. If $f: A \times B \rightarrow B$ has type $(e, 1_B)$ then $(\beta, \beta h(f))^\alpha = (\beta, -[-\alpha, -\beta]_f)$.

5. The cofiber theorem. The principal example of a twisted Whitehead product is provided by a fibration over a suspension. In this section we extend the cofiber theorem [1, Theorem 4.2] to this class of examples.

Suppose $p: E \rightarrow \Sigma X$ is a (based) Hurewicz fibration over a suspension space with fiber F . Let $\gamma: X \times F \rightarrow F$ be a clutching function for p . The class of γ is uniquely determined by p and so we write

$$W_p: X * F \rightarrow \Sigma X \vee \Sigma F \quad (\text{rather than } W_\gamma).$$

We denote by $p * \epsilon^0$ the fiberwise join of p and the projection $\Sigma X \times S^0 \rightarrow \Sigma X$. Thus $E(p * \epsilon^0)$ is the unreduced double mapping cylinder on p with itself. Since spaces are well pointed we may (without changing the homotopy type) regard $E(p * \epsilon^0)$ as the reduced double mapping cylinder. Doing this gives an inclusion $\Sigma X \vee \Sigma F \subset E(p * \epsilon^0)$.

(5.1) THEOREM. The homotopy commutative diagram

$$\begin{array}{ccc} X * F & \xrightarrow{W_p} & \Sigma X \vee \Sigma F \\ \downarrow & & \cap \\ * & \rightarrow & E(p * \epsilon^0) \end{array}$$

is a homotopy pushout. In particular $C_{W_p} \simeq E(p * \epsilon^0)$.

PROOF. This follows from a direct application of Theorem 1.3 in [5]. $\square$

(5.2) EXAMPLES. (i) For the projection fibration $\text{pr}: \Sigma A \times B \rightarrow \Sigma A$ with fiber B , W_{pr} is the composite

$$A * B \xrightarrow{W} \Sigma A \vee \Sigma B \xrightarrow{r \vee r} \Sigma A \vee \Sigma B$$

and $E(p * \epsilon^0) \simeq \Sigma A \times \Sigma B$. This recovers the classical cofiber theorem $C_W \simeq \Sigma A \times \Sigma B$ [1, Theorem 4.2].

(ii) Let $p: V_{n+1,k+1} \rightarrow S^n$ be the Stiefel fibration with fiber $V_{n,k}$. For $k = 1$, $C_{W_p} \simeq S^n \times S^n$ and $W_p \simeq W: S^{2n-1} \rightarrow S^n \vee S^n$. This reflects the fact that $p * \epsilon^0$ is fiber homotopically trivial. Thus we obtain an alternative definition of the classical Whitehead product for spheres. However, for $k > 1$, $W_p \neq W: S^{n-1} * V_{n,k} \rightarrow S^n \vee \Sigma V_{n,k}$.

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