SIMPLE NEAR-RING CENTRALIZERS OF FINITE RINGS

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ABSTRACT. For a finite ring R with identity and a finite unital R -module V we call $C(R) = \{f: V \rightarrow V | f(\alpha v) = \alpha f(v) \text{ for all } \alpha \in R, v \in V\}$ the near-ring centralizer of R . We investigate the structure of $C(R)$ and obtain a characterization of those rings R for which $C(R)$ is a simple nonring.

1. Introduction. Let H be a commutative ring with 1 and let V be a finitely generated H -module with $S = \text{End}_H(V)$. If H is a field and if R is a simple subring of S containing H then it is well known that the centralizer of R in S , $C_S(R)$, is a simple ring. However if $C_S(R)$ is simple it is easy to see that R need not be simple. As far as the authors know, a characterization of the subrings R of S such that $C_S(R)$ is simple is an unsolved problem. In this paper we consider a nonlinear analogue. In particular let S denote the near-ring $S = \text{Map}_H(V) = \{f: V \rightarrow V | f(\alpha v) = \alpha f(v) \text{ for all } \alpha \in H, v \in V\}$. Let R be a subring of S of H -endomorphisms of V and assume R contains H . The set $C(R) = \{f \in S | f(rv) = rf(v) \text{ for all } r \in R, v \in V\}$ is a near-ring, called the near-ring centralizer of R in S . It is the goal of this paper to consider the following problems in the case where H is a finite commutative ring with 1: If R is a simple ring is $C(R)$ a simple near-ring? If $C(R)$ is a simple near-ring what can be said about R ? Which simple nonrings (near-rings that are not rings) have the form $C(R)$ for some subring R of S ?

The above questions arise naturally from a different point of view. It has been shown by Betsch [1] that if N is a finite simple near-ring which is not a ring then there exists a finite group G and a fixed point free group of automorphisms Γ of G such that $N \cong C_\Gamma(G) \equiv \{f: G \rightarrow G | f(\gamma a) = \gamma f(a) \text{ for all } \gamma \in \Gamma, a \in G\}$. As long as G is a group and Γ is a semigroup of endomorphisms of G then $C_\Gamma(G)$ forms a near-ring. The structure of $C_\Gamma(G)$ for various G 's and Γ 's has been investigated in [4], [5] and [6]. In this paper we study the structure of $C_\Gamma(G)$ when Γ is a finite ring with 1 and G is a unitary Γ -module. In this setting we investigate the problems mentioned above.

2. Simple centralizers. For the rest of this paper R will denote a finite ring with 1 and V a finite unital R -module. The near-ring centralizer of R is $C(R) = \{f: V \rightarrow V | f(rv) = rf(v) \text{ for all } r \in R, v \in V\}$.

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THEOREM 1. *Let R be a finite simple ring and let V be a finite unital R -module. Then $C(R)$ is a simple near-ring. Moreover $C(R)$ is a nonring if and only if R is a field and $\dim_R V > 1$.*

PROOF. Let $\{e_{ij}\}$ be a set of matrix units for R where $1 = e_{11} + \cdots + e_{tt}$, the e_{ii} primitive idempotents. The R -module V forms a vector space over the center F of R . We have $V = V_1 \oplus \cdots \oplus V_t$ where $V_i = e_{ii}V$, a vector space direct sum. For $f \in C(R)$, $v_i \in V_i$ we have $f(v_1 + \cdots + v_t) = v'_1 + \cdots + v'_t$ for some $v'_i \in V_i$. Since $e_{ii}f = fe_{ii}$ we have $f(v_i) = v'_i$ and so $f(v_1 + \cdots + v_t) = f(v_1) + \cdots + f(v_t)$.

If $t = 1$ then R is a field and $C(R)$ is well known to be simple [1]. Also $C(R)$ is a nonring when $\dim_R V > 1$. (For if $v \neq 0 \in V$ define $f: V \rightarrow V$ by $f(\alpha v) = \alpha v$, $\alpha \in R$, and $f(x) = 0$ otherwise. Then $f \in C(R)$ and f is not linear.)

Assume now that $t > 1$. To show $f \in C(R)$ is a linear transformation on V it suffices to show f is linear on each V_i . Without loss of generality we choose $i = 1$. For $v_1 \in V_1$, $v_2 \in V_2$ we have

$$\begin{aligned} (1 + e_{12})f(v_1 + v_2) &= (1 + e_{12})(f(v_1) + f(v_2)) \\ &= f(v_1) + f(v_2) + f(e_{12}v_2), \end{aligned}$$

and

$$\begin{aligned} f((1 + e_{12})(v_1 + v_2)) &= f(v_1 + v_2 + e_{12}v_2) \\ &= f(v_1 + e_{12}v_2) + f(v_2). \end{aligned}$$

Hence $f(v_1 + e_{12}v_2) = f(v_1) + f(e_{12}v_2)$. Since $e_{12}V_2 = V_1$, f is linear on V_1 . Thus $C(R)$ is a ring and is simple by the double centralizer theorem for rings [3, p. 132].

COROLLARY. *Assume R is a finite semisimple ring and V is a finite unital R -module. If none of the simple summands of R is a field then $C(R)$ is a ring.*

Assume now that $C(R)$ is a simple near-ring. From the Betsch characterization of simple near-rings [1] it is easy to see that the nonzero elements of the center of $C(R)$ are invertible. Since the center of R is contained in the center of $C(R)$, the center of R is a field F . Hence R is an algebra over F , V is a vector space over F , and R may be viewed as a ring of linear transformations acting on V .

The Wedderburn principal theorem is valid for R [7, p. 164], that is $R = S + N$, a vector space direct sum, where N is the radical of R and S is semisimple with $R/N \cong S$.

If $N = (0)$ then R must be simple since its center is a field, and Theorem 1 applies. Hence we may assume $N \neq (0)$. In the following sequence of lemmas we show that if R is not a field then $C(R)$ is a ring.

LEMMA 1. *If $C(R)$ is simple with $R = S + N$, $N \neq (0)$, then S is not a simple ring.*

PROOF. Suppose $N^k = (0)$ with $N^{k-1} \neq (0)$. Let $W_1 = \ker N^{k-1} = \{v \in V \mid nv = 0 \text{ for all } n \in N^{k-1}\}$, a proper subspace of V and an S -submodule of V . As an S -module V is completely reducible so $V = W_1 \oplus W_2$, an S -module direct sum. Select $w_2 \neq 0$ in W_2 ; then Sw_2 is a direct summand of W_2 and $V = W_1 \oplus Sw_2 \oplus \bar{W}_2$. The S -annihilator of w_2 , $\text{ann}(w_2)$, is a left ideal of S so $S = \text{ann}(w_2) \oplus S\bar{e}$ as left ideals, where $\bar{e}$ is an idempotent. If $\bar{e}N^{k-1} \neq (0)$ then there exists a $n_0 \in N^{k-1}$ such that $\bar{e}n_0 \neq 0$. Select $v \in V$ with $\bar{e}n_0(v) \neq 0$. Every element of V can be uniquely written in the form $w_1 + \bar{s}w_2 + \bar{w}_2$, where $w_1 \in W_1$, $\bar{s} \in S\bar{e}$ and $\bar{w}_2 \in \bar{W}_2$. Define $f: V \rightarrow V$ by $f(w_1 + \bar{s}w_2 + \bar{w}_2) = \bar{s}\bar{e}n_0(v)$. The function f is nonzero since $f(\bar{e}w_2) = \bar{e}n_0(v) \neq 0$, and since $nW_2 \subseteq W_1$ for all $n \in N$ we have $f \in C(R)$. Also the set $I = \{g \in C(R) \mid g(W_1) = \{0\}\}$ is an ideal of $C(R)$ containing f , which is a contradiction to the simplicity of $C(R)$, so $\bar{e}N^{k-1} = (0)$. Thus the S -annihilator of N^{k-1} is a proper ideal of S .

LEMMA 2. *Assume $C(R)$ is simple and not a ring. Then S is a direct sum of fields.*

PROOF. We have $R = S + N$ where $S = S_1 \oplus S_2 \oplus \cdots \oplus S_t$, S_i simple and $t > 1$. Suppose S_1 is not a field. Let e_i be the identity for S_i . We have $V = e_1V \oplus \cdots \oplus e_tV$ where each e_iV is $C(R)$ -invariant. Since the simple ring S_1 acts on e_1V the map $\Phi: C(R) \rightarrow C(S_1)$ given by $f \rightarrow \bar{f}$, where $\bar{f}$ is f restricted to e_1V , is an imbedding. Since $C(S_1)$ is a ring, so is $C(R)$. Thus S must be a direct sum of fields.

LEMMA 3. *Suppose $C(R)$ is simple and that in R , $1 = e_1 + e_2$ where e_1 and e_2 are orthogonal idempotents that belong to the center of S . Let $V = W_1 \oplus W_2$ where $W_i = e_iV$. Assume there exists a positive integer l such that $\ker N^l \subseteq W_1$ and $\ker N^{l+1} \cap W_2 \neq (0)$. Then $C(R)$ is a ring.*

PROOF. Let $K = \ker N^{l+1} \cap W_2$, an S -submodule of V which is $C(R)$ -invariant. The subspace $\langle N^lK \rangle$ of V is also an S -submodule, and $\langle N^lK \rangle \subseteq \ker N \subseteq \ker N^l \subseteq W_1$. As S -modules we have $V = \bar{W}_1 \oplus \langle N^lK \rangle \oplus K \oplus \bar{W}_2$ where $W_1 = \bar{W}_1 \oplus \langle N^lK \rangle$, $W_2 = K \oplus \bar{W}_2$.

We show now that if $f \in C(R)$ then f is linear on $\langle N^lK \rangle \oplus K$. For $v_1, v_2 \in K$, $n_1, n_2 \in N^l$ we have $n_i v_i \in \langle N^lK \rangle$. Also $(1 + n_1)f(v_1 + n_2v_2) = f((1 + n_1)(v_1 + n_2v_2))$ which implies that $f(n_1v_1 + n_2v_2) = f(n_1v_1) + f(n_2v_2)$. For $n \in N^l$, $v_1, v_2 \in K$ we have, from above, that $n(f(v_1 + v_2) - f(v_1) - f(v_2)) = 0$. This means $f(v_1 + v_2) - f(v_1) - f(v_2) \in \ker N^l \cap W_2 = (0)$ and so f is linear on K .

For $n_i \in N^l$, $v_i \in K$, $i = 1, 2, 3$, we have

$$(1 + n_1)f(v_1 + n_2v_2 + n_3v_3) = f((1 + n_1)(v_1 + n_2v_2 + n_3v_3)),$$

so

$$f(n_1v_1 + n_2v_2 + n_3v_3) = f(n_1v_1) + f(n_2v_2) + f(n_3v_3).$$

Continuing we get f is linear on $\langle N'K \rangle$, and hence linear on $\langle N'K \rangle \oplus K$. Moreover since K is $C(R)$ -invariant so is $\langle N'K \rangle$ and hence $\langle N'K \rangle \oplus K$ is both $C(R)$ and S -invariant. Let

$$C(S; \langle N'K \rangle \oplus K) = \{ f: \langle N'K \rangle \oplus K \rightarrow \langle N'K \rangle \oplus K \mid f(\alpha v) = \alpha f(v), \\ \alpha \in S, v \in \langle N'K \rangle \oplus K \}.$$

The map given by $f \rightarrow \bar{f}$, where $\bar{f}$ is f restricted to $\langle N'K \rangle \oplus K$, is an imbedding of $C(R)$ into $C(S; \langle N'K \rangle \oplus K)$. Since the range of this imbedding is a ring, $C(R)$ is also a ring.

THEOREM 2. *Suppose R is a finite ring with 1, not a field, and V is a finite unital R -module. If $C(R)$ is simple then it is a ring.*

PROOF. From Lemmas 1 and 2 we have $R = S + N$ where S is a direct sum of at least two fields and $N^k = (0)$, $N^{k-1} \neq (0)$. So $S = Se_1 \oplus \cdots \oplus Se_t$ where each e_i is a primitive idempotent, and $V = W_1 \oplus \cdots \oplus W_t$ where $W_i = e_i V$.

As S -modules we have $V = \ker N^{k-1} \oplus \bar{W}$. If $\bar{w} \neq 0 \in \bar{W}$ then $\bar{w} = w_1 + \cdots + w_t$, $w_i \in W_i$, and we may assume $w_t \neq 0$. Then $e_t \bar{w} = w_t \in \bar{W} \cap W_t$. We show now that $\ker N \cap W_t = (0)$. Assume $k \neq 0$ belongs to $\ker N \cap W_t$. As S -modules

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_{t-1} \oplus \bar{S}w_t \oplus \bar{W}_t$$

where $\bar{S} = Se_t$ and $W_t = \bar{S}w_t \oplus \bar{W}_t$. Define $f: V \rightarrow V$ by

$$f(w_1 + w_2 + \cdots + w_{t-1} + \bar{S}w_t + \bar{w}_t) = \bar{S}k.$$

Since $nV \subseteq \ker N^{k-1}$ for all $n \in N$ and since $\bar{S}w_t \cap \ker N^{k-1} = (0)$, we have $f \in C(R)$. Also $f \neq 0$ since $f(w_t) = e_t k = k \neq 0$. Moreover $f(\ker N^{k-1}) = (0)$ and this contradicts the simplicity of $C(R)$. Hence $\ker N \cap W_t = (0)$.

We now have $\ker N \subseteq W_1 \oplus W_2 \oplus \cdots \oplus W_{t-1}$ with $\ker N \cap W_t = (0)$. Let $l \geq 1$ be such that $\ker N^l \subseteq W_1 \oplus \cdots \oplus W_{t-1}$ but $\ker N^{l+1} \not\subseteq W_1 \oplus \cdots \oplus W_{t-1}$. Then there exists a $v \in \ker N^{l+1}$ with $v = w_1 + \cdots + w_{t-1} + w_t$ and $w_t \neq 0$. This means $e_t v = w_t$ is a nonzero element in $\ker N^{l+1} \cap W_t$. By Lemma 3, $C(R)$ is a ring.

Under the conditions of the above theorem, the simple near-ring $C(R)$ is a simple ring which clearly contains $\text{End}_R(V)$, the ring of R -endomorphisms of V . We now show that $C(R)$ is in fact equal to the ring $\text{End}_R(V)$. This is established by showing that the near-ring module V is indeed a ring module over $C(R)$.

THEOREM 3. *Suppose R is a finite ring with 1, not a field, and V is a finite unital R -module. If $C(R)$ is simple then $C(R) = \text{End}_R(V)$, the ring of R -endomorphisms of V .*

PROOF. From Theorem 2 $C(R)$ is a ring. We need to show $C(R) = \text{End}_R(V)$. Clearly V is a near-ring module over the simple ring $C(R)$. Since

$C(R)$ is a ring it is a distributively generated near-ring, so submodules of V and $C(R)$ -subgroups of V coincide [8, p. 174]. We have $C(R) = A_1 \oplus \cdots \oplus A_i$ where the A_i are minimal left ideals of $C(R)$. For $v \in V$ either $A_i \cong A_i v$ or $A_i v = (0)$. Hence $V = \sum_{v \in V} \sum_i A_i v$, a sum of irreducible submodules of V . But then V is a direct sum of irreducible submodules $V = V_1 \oplus \cdots \oplus V_s$. Since each V_i is a cyclic submodule of V it is clear that $f \in C(R)$ is additive on each V_i . Using the definition of a near-ring module it is easy to see that $f \in C(R)$ is additive on all of V . Thus V is a ring module and $C(R)$ is the complete ring of R -endomorphisms of V as desired.

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