A CHARACTERIZATION OF MAXIMALLY ALMOST PERIODIC GROUPS

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ABSTRACT. It is proved that a topological group G is maximally almost periodic if and only if G admits an action π on a compact Hausdorff space X such that the transformation group (X, G, π) is equicontinuous and effective. Using this characterization, it is proved that if H is a closed uniform subgroup of a topological group G , then G is maximally almost periodic if and only if H is maximally almost periodic. The latter gives as corollaries the results of Kuranishi, Murakami, Grosser and Moskowitz concerning maximally almost periodic groups.

1. Definitions and the main result. A topological group G is said to be maximally almost periodic (MAP) provided that the set of complex-valued continuous almost periodic functions defined on G separates the points of G . In this paper, contrary to customary usage, the function or transformation sign is placed on the right. That is, if X and Y are sets, ϕ denotes a transformation of X into Y and $x \in X$, then $x\phi$ denotes the unique element of Y determined by x and ϕ . A transformation group (X, G, π) is said to be effective provided that if $t \in G$ such that $t \neq e$, where e is the identity element of G , then there exists $x \in X$ such that $(x, t)\pi = xt \neq x$. Let X be a topological space, let Y be a uniform space with uniform structure $\mathcal{U}$ and let E be a set of mappings of X into Y . If $x \in X$, then E is said to be equicontinuous at x provided that if $\alpha \in \mathcal{U}$, then there exists a neighborhood U of x such that $y \in U$ and $\phi \in E$ implies $(y\phi, x\phi) \in \alpha$. The set E is said to be equicontinuous provided that E is equicontinuous at every point of X . Let (X, G, π) be a transformation group with a uniform space X . If $t \in G$, then the t -transition of (X, G, π) , denoted by π' , is the mapping $\pi': X \rightarrow X$ such that $x\pi' = (x, t)\pi = xt$ ($x \in X$). The set $\{\pi' | t \in G\}$ is called the transition group of (X, G, π) . If $x \in X$, then the x -motion of (X, G, π) , denoted by π_x , is the mapping $\pi_x: G \rightarrow X$ such that $t\pi_x = (x, t)\pi = xt$ ($t \in G$). The transformation group (X, G, π) is said to be equicontinuous provided that the transition group $\{\pi' | t \in G\}$ of (X, G, π) is a set of equicontinuous mappings of X into X . All topological groups in the paper are assumed to be Hausdorff and for a general reference on the theory of topological transformation

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groups, we consult Gottschalk and Hedlund [1].

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In this section, we shall prove the following main result of the paper:

(1.1) THEOREM. *Let G be a topological group. Then the following statements are equivalent:*

- (1) G is MAP;
- (2) G admits an action π on a compact Hausdorff space X such that the transformation group (X, G, π) is equicontinuous and effective.

PROOF. Assume (1). We prove (2). It is well known that there is a continuous monomorphism ϕ of G into a compact group $\mathfrak{M}$. Define $\pi: \mathfrak{M} \times G \rightarrow \mathfrak{M}$ by $(x, t)\pi = x(t\phi)(x \in \mathfrak{M}, t \in G)$. Then it is easy to verify that $(\mathfrak{M}, G, \pi)$ is an effective transformation group. In order to show that $(\mathfrak{M}, G, \pi)$ is equicontinuous, let $x \in \mathfrak{M}$, let $\mathcal{U}$ be the right uniform structure on $\mathfrak{M}$ (which is equivalent to the left uniform structure on $\mathfrak{M}$), and let $\alpha \in \mathcal{U}$. Then there is a neighborhood V of the identity element of $\mathfrak{M}$ such that $y \in \mathfrak{M}$ and $yx^{-1} \in V$ implies $(y, x) \in \alpha$. Set $U = Vx$. Then U is a neighborhood of x . Now, if $y \in U$ and $t \in G$, then

$$(y\pi')(x\pi')^{-1} = (y(t\phi))(x(t\phi))^{-1} = (y(t\phi))((t\phi)^{-1}x^{-1}) = yx^{-1} \in V$$

and hence $(y\pi', x\pi') \in \alpha$. Therefore $(\mathfrak{M}, G, \pi)$ is equicontinuous. This shows that (1) implies (2).

Assume (2). We prove (1). Let $t, t' \in G$ such that $t \neq t'$. We want to show that there exists a complex-valued continuous almost periodic function f defined on G such that $tf \neq t'f$. Since (X, G, π) is effective, there exists $x \in X$ such that $xt \neq xt'$, or equivalently, $t\pi_x \neq t'\pi_x$. Let h be a complex-valued continuous function defined on X such that $(t\pi_x)h \neq (t'\pi_x)h$. Now, since (X, G, π) is equicontinuous and X is compact, (X, G, π) is uniformly equicontinuous, that is, the transition group $\{\pi'|t \in G\}$ is a set of uniformly equicontinuous mappings of X into X , and thus, by Theorem [1, 4.38, p. 37], (X, G, π) is almost periodic. By Theorem [1, 4.65, p. 43], the function $f = \pi_x \circ h$ is a complex-valued continuous bilaterally uniformly almost periodic function defined on G and hence is continuous almost periodic such that $tf \neq t'f$. This proves that (2) implies (1).

The proof is completed.

(1.2) REMARKS. Theorem 1.1 gives a characterization of the acting groups of the class of all equicontinuous and effective transformation groups with compact Hausdorff phase spaces on the one hand and it provides a new way to study the structures of MAP groups on the other. In view of a structure theorem of Freudenthal and Weil, if the acting group of an equicontinuous and effective transformation group with a compact Hausdorff space is connected and locally compact, then by Theorem 1.1, it is a direct product of a compact group and a vector group. In Huang [4], a simple proof that every locally compact MAP group is unimodular due to Leptin and Robertson [6]

was given by making use of a partial result of Theorem 1.1.

2. Uniform subgroups. A subgroup H of a topological group G is said to be uniform provided that there is a compact subset K of G such that $G = KH$ (or equivalently, $G = HK$). In this section, we shall use Theorem 1.1 to prove the following:

(2.1) **THEOREM.** *Let G be a topological group and let H be a closed uniform subgroup of G . Then G is MAP if (and only if) H is MAP.*

PROOF. By Theorem 1.1, H admits an action π on a compact Hausdorff space X such that the transformation group (X, H, π) is equicontinuous and effective. Define $\sigma: (X \times G) \times (H \times G) \rightarrow X \times G$ by

$$\begin{aligned} ((x, g_0), (h, g))\sigma &= (x, g_0)(h, g) = ((x, h)\pi, h^{-1}g_0g) \\ &= (xh, h^{-1}g_0g) \quad (x \in X, h \in H \text{ and } g_0, g \in G). \end{aligned}$$

Then $(X \times G, H \times G, \sigma)$ is a transformation group. Let $Y = X \times G/H \times \{e\}$, where e is the identity element of G , denote the orbit partition $\{(x, g_0)H \times \{e\} | (x, g_0) \in X \times G\}$ under $H \times \{e\}$ provided with its partition topology. Then, by a recent result of Gottschalk [2], Y is compact and Hausdorff. Define $\mu: Y \times (\{e\} \times G) \rightarrow Y$ by

$$((x, g_0)H \times \{e\}, (e, g))\mu = (x, g_0g)H \times \{e\}.$$

Then, by a similar argument to the one given above, one can show that $(Y, \{e\} \times G, \mu)$ is an equicontinuous transformation group. In order to show that $(Y, \{e\} \times G, \mu)$ is effective, let $(e, g) \in \{e\} \times G$ such that

$$((x, g_0)H \times \{e\}, (e, g))\mu = (x, g_0)H \times \{e\}$$

$((x, g_0)H \times \{e\} \in Y)$. Then there exist $h, h' \in H$ such that $(xh, h^{-1}g_0g) = (xh', h'^{-1}g_0)$. Since (X, H, π) is effective, $h = h'$ and thus $g = e$. Therefore $(Y, \{e\} \times G, \mu)$ is effective. Now, since $(Y, \{e\} \times G, \mu)$ is equicontinuous and effective, by Theorem 1.1, $\{e\} \times G$ is MAP and consequently, G is MAP.

The proof is completed.

(2.2) **REMARKS.** Kuranishi [5] and Murakami [8] proved that a locally compact group G such that G/G_0 is compact, where G_0 is the identity component of G , is MAP if and only if G_0 is MAP. Grosser and Moskowitz [3] defined the notion of central topological groups and proved that every locally compact central topological group is MAP. Their proofs involved either Lie groups, the theorem of Gelfand-Raikov or the structure theorem of central topological groups. Theorem 2.1 provides a direct and unified proof of these results concerning MAP groups.

3. Corollaries. As some immediate consequences of Theorem 2.1, we have

COROLLARY 3.1 (KURANISHI, MURAKAMI). *Let G be a locally compact group such that G/G_0 is compact, where G_0 is the identity component of G . Then G is MAP if and only if G_0 is MAP.*

PROOF. It is clear that G_0 is a closed uniform subgroup of G and the result follows from Theorem 2.1.

The proof is completed.

Let G be a locally compact group and let $Z(G)$ be its center. Then $Z(G)$ is a locally compact abelian group and hence it is MAP. Furthermore, if $G/Z(G)$ is compact, then $Z(G)$ is a (closed) uniform subgroup of G . Thus, we have

COROLLARY 3.2 (GROSSER AND MOSKOWITZ). *Every locally compact central topological group is MAP.*

Robertson [9] defined the notion of Moore groups, see also Moore [7], and gave the following characterization: G is a Moore group if and only if G contains a characteristic subgroup H such that H has finite index in G and H is a Takahashi group, that is, H is MAP and the derived group H' of H has compact closure. Even though the results of Corollaries 3.1 and 3.2 are also consequences of the fact that every Moore group is MAP, yet the latter fact depends on the theorem of Gelfand-Raikov. The following is a direct consequence of Robertson's characterization of Moore groups and Theorem 2.1:

COROLLARY 3.3. *Every Moore group is MAP.*

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