ON TWO QUESTIONS OF HALMOS CONCERNING SUBSPACE LATTICES

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ABSTRACT. An example is constructed of a nonreflexive pentagonal lattice of subspaces. It follows that reflexivity is not invariant under lattice isomorphism, even for finite lattices.

Following Halmos [2], [3] we say that a lattice $\mathcal{L}$ of subspaces of a Hilbert space is *reflexive* if there is a collection of bounded linear operators whose lattice of invariant subspaces is $\mathcal{L}$. The questions of Halmos [3] referred to in the title are:

- (i) Is every subspace lattice which is (isomorphic to) the pentagon reflexive?
- (ii) If two finite subspace lattices are isomorphic and one is reflexive, must the other be reflexive?

In this note we exhibit a nonreflexive pentagon, giving a negative answer to the first question. Since Halmos [3] has given an example of a reflexive pentagon, this shows that the second question also has a negative answer.

A nonreflexive pentagon can be described as follows. Let K denote the diagonal operator on l^2 defined by $K\{\alpha_n\} = \{\alpha_n/n\}$, and define $x = \{1/n\}$ and $y = \{1/n^{3/2}\}$. Now let the Hilbert space $\mathcal{H}$ be $l^2 \oplus l^2$ and let $\mathcal{L}$ consist of $\{0\}$, $l^2 \oplus \{0\}$, $\mathcal{G} = \{z \oplus Kz : z \in l^2\}$, $\mathcal{G}_0 = \{z \oplus (Kz + \alpha x + \beta y) : z \in l^2, \alpha \in \mathbb{C}, \beta \in \mathbb{C}\}$, and $\mathcal{H}$. We must check that $\mathcal{L}$ is a lattice with Hasse diagram:

All the relations of the above diagram are immediate except that $\mathcal{G}_0 \cap (l^2 \oplus \{0\}) = \{0\}$. But if $Kz + \alpha x + \beta y = 0$ then $\alpha x + \beta y$ is in the range of K . Hence $\sum_{n=1}^{\infty} n^2 |\alpha/n + \beta/n^{3/2}|^2 < \infty$, so $\alpha = \beta = 0$. Thus $Kz = 0$, and $z = 0$. It follows that $\mathcal{G}_0 \cap (l^2 \oplus \{0\}) = \{0\}$.

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THEOREM. *The lattice $\mathcal{L}$ is not reflexive.*

PROOF. Let $\mathcal{Q}$ denote the algebra of all operators leaving the elements of $\mathcal{L}$ invariant. It is apparent (cf. [2], [3]) that $\mathcal{L}$ is reflexive only if every invariant subspace of $\mathcal{Q}$ is an element of $\mathcal{L}$. We show that $\mathfrak{N} = \{z \oplus (Kz + \lambda y) : z \in l^2, \lambda \in \mathbb{C}, K \text{ and } y \text{ as above}\}$, which is not an element of $\mathcal{L}$, is invariant under every operator in $\mathcal{Q}$.

Let $T \in \mathcal{Q}$. With respect to the decomposition $l^2 \oplus l^2$ of $\mathcal{H}$, T has the form $T = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$. The invariance of $\mathfrak{G}$ under T yields $KA + KBK = DK$. For convenience let $C = D - KB$; then $KA = CK$. Since T leaves $\mathfrak{G}_0$ invariant, $T(0 \oplus y)$ is in $\mathfrak{G}_0$ and so is $T(0 \oplus y) - (By \oplus KBy) = 0 \oplus Cy$. Thus $Cy = \alpha x + \beta y$ for some α and β . Note that $CK = KA$ implies that C leaves the range of K invariant. Hence, by Corollary 2 of [6], C also leaves the range $\mathfrak{R}$ of $K^{1/2}$ invariant. Now $y = K^{1/2}x$ is in $\mathfrak{R}$, hence so is Cy and so is αx . But $x \notin \mathfrak{R}$. Therefore $\alpha = 0$ and $Cy = \beta y$.

The invariance of $\mathfrak{N}$ under T is now easily seen. For if $z \oplus (Kz + \gamma y) \in \mathfrak{N}$, then

$$\begin{aligned} \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \begin{pmatrix} z \\ Kz + \gamma y \end{pmatrix} &= \begin{pmatrix} Az + BKz + \gamma By \\ (C + KB)(Kz + \gamma y) \end{pmatrix} \\ &= \begin{pmatrix} Az + BKz + \gamma By \\ KAz + \gamma \beta y + KBKz + \gamma KBy \end{pmatrix} = \begin{pmatrix} u \\ Ku + \gamma \beta y \end{pmatrix}, \end{aligned}$$

where $u = Az + BKz + \gamma By$. Hence $\mathfrak{N}$ is invariant under T .

REMARKS. (a) Instead of quoting Corollary 2 of [6] in the above proof, we could use the more general Proposition 5 of [1], once the proof of Proposition 5 is modified; see the remarks following Corollary 2 of [6].

(b) For finite lattices of subspaces of finite-dimensional spaces, the answer to question (ii) above is affirmative. A theorem of Johnson [5] implies that such a lattice is reflexive if and only if it is distributive.

(c) There are many known results concerning reflexive lattices: see [4] for other references to recent work.

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